Formalization of Foundations of Geometry

An overview of the GeoCoq library

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March 2017, Argo Group Seminar

GeoCoq

- An Open Source library about foundations of geometry
- Written by Gabriel Braun, Pierre Boutry, Charly Gries and Julien Narboux
- License: LGPL3
- 3500 lemmas, 130kloc

Foundations of geometry

1. Synthetic geometry
2. Analytic geometry
3. Metric geometry
4. Transformations based approaches

Synthetic approach

Assume some undefined geometric objects + geometric predicates + axioms . . .
The name of the assumed types are not important.

- Hilbert's axioms:

types: points, lines and planes
predicates: incidence, between, congruence of segments, congruence of angles

- Tarski's axioms:

types: points
prédicats: between, congruence

- . . . many variants

Notions primitives

When we set out to construct a given discipline, we distinguish, first of all, a certain small group of expressions of this discipline that seem to us to be immediately understandable; the expressions in this group we call PRIMITIVE TERMS or UNDEFINED TERMS, and we employ them without explaining their meanings. At the same time we adopt the principle: not to employ any of the other expressions of the discipline under consideration, unless its meaning has first been determined with the help of primitive terms and of such expressions of the discipline whose meanings have been explained previously. The sentence which determines the meaning of a term in this way is called a DEFINITION,... Alfred Tarski, Introduction to Logic: and to the Methodology of Deductive Sciences, p 118

Example of books using a synthetic approach:

- Euclide (1998). *Les Éléments*. *Les Éléments*
- David Hilbert (1899). *Grundlagen der Geometrie*. *Grundlagen der Geometrie*
- Borsuk and Szmielew: *Foundations of Geometry*
- Robin Hartshorne (2000). *Geometry : Euclid and beyond*. Undergraduate texts in mathematics *Geometry: Euclid and Beyond*
- Marvin J. Greenberg (1993). *Euclidean and Non-Euclidean Geometries - Development and History*. *Euclidean and non-euclidean Geometries, Development and History*
- Specht *et. al.*: *Euclidean Geometry and its Subgeometries*

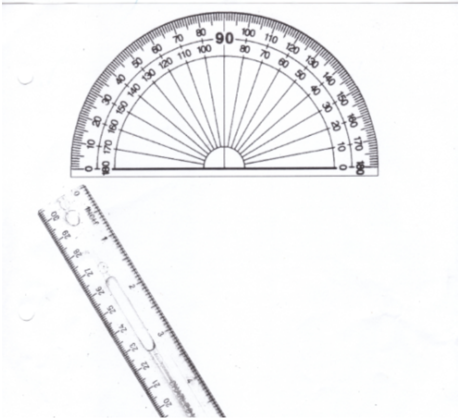
Analytic approach

We assume we have numbers (a field $\mathbb{F}$).
We define geometric objects by their coordinates.
Points := $\mathbb{F}^n$

Metric approach

Compromise between synthetic and metric approach.
We assume both:

- numbers (a field)
- geometric objects
- axioms



- Birkhoff's axioms: points, lines, reals, ruler and protractor
- Chou-Gao-Zhang's axioms: points, numbers, three geometric quantities

Examples of books using metric approach:

- E.E. Moise (1990). *Elementary Geometry from an Advanced Standpoint*.
- Richard S Millman and George D Parker (1991). *Geometry, A Metric Approach with Models*.

Transformation groups

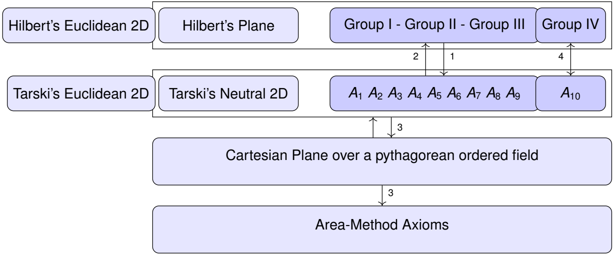
Erlangen program. Foundations of geometry based on group actions and invariants.

Felix Klein

Comparison

	Synthetic	Analytic
Logical Reasoning	☺	☹
Proof reuse between geometries	☺	☹
Computations	☹	☺
Automatic proofs	☹	☺

Overview of the axiom systems



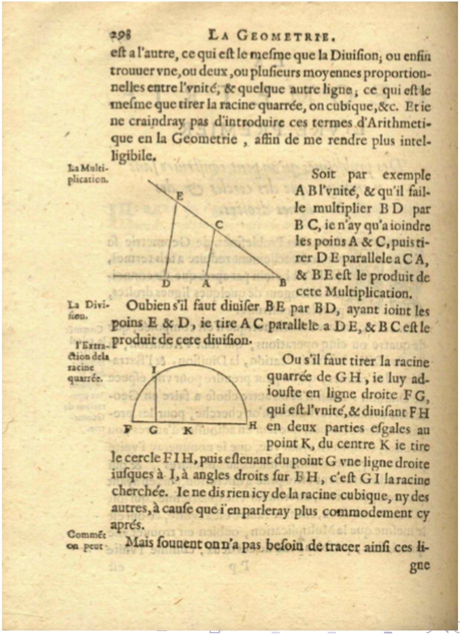
[1] Gabriel Braun, Pierre Boutry, and Julien Narboux (2016). "From Hilbert to Tarski". In: *Eleventh International Workshop on Automated Deduction in Geometry*. Proceedings of ADG 2016

[2] Gabriel Braun and Julien Narboux (2012). "From Tarski to Hilbert". English. In: *Post-proceedings of Automated Deduction in Geometry 2012*. Vol. 7993. LNCS

[3] Pierre Boutry, Gabriel Braun, and Julien Narboux (2017). "Formalization of the Arithmetization of Euclidean Plane Geometry and Applications". In: *Journal of Symbolic Computation*

[4] Pierre Boutry, Julien Narboux, and Pascal Schreck (2015). "Parallel postulates and decidability of intersection of lines: a mechanized study within Tarski's system of geometry".

Arithmetization of Geometry



est a l'autre, ce qui est le mesme que la Diuision; ou enfin trouuer vne, ou deux, ou plusieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne; ce qui est le mesme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligible.

La Multiplication.

Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E est le produit de cete Multiplication.

La Diuision.

Oubien s'il faut diuiser B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C est le produit de cete diuision.

l'Extraction dela racine quarrée.

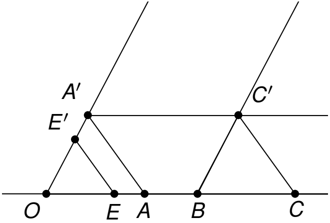
Ou s'il faut tirer la racine quarrée de G H, ie luy adiouste en ligne droite F G, qui est l'vnité, & diuisant F H en deux parties esgales au point K, du centre K ie tire le cercle F I H, puis esleuant du point G vne ligne droite iusques à I, à angles droits sur F H, c'est G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cause que i'en parleray plus commodement cy aprés.

Commét on peut

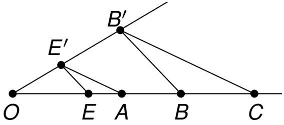
Mais souuent on n'a pas besoin de tracer ainsi ces li-
gne

René Descartes (1925). *La géométrie*.

Addition



Multiplication



Characterization of geometric predicates

Geometric predicate	Characterization
$AB \equiv CD$	$(x_A - x_B)^2 + (y_A - y_B)^2 - (x_C - x_D)^2 + (y_C - y_D)^2 = 0$
Bet $A\ B\ C$	$\exists t, 0 \le t \le 1 \wedge t(x_C - x_A) = x_B - x_A \wedge t(y_C - y_A) = y_B - y_A$
$Col\ A\ B\ C$	$(x_A - x_B)(y_B - y_C) - (y_A - y_B)(x_B - x_C) = 0$
I midpoint of AB	$2x_I - (x_A + x_B) = 0 \wedge 2y_I - (y_A + y_B) = 0$
$PerABC$	$(x_A - x_B)(x_B - x_C) + (y_A - y_B)(y_B - y_C) = 0$
$AB \parallel CD$	$(x_A - x_B)(x_C - x_D) + (y_A - y_B)(y_C - y_C) = 0 \wedge (x_A - x_B)(x_A - x_B) + (y_A - y_B)(y_A - y_B) \neq 0 \wedge (x_C - x_D)(x_C - x_D) + (y_C - y_D)(y_C - y_D) \neq 0$
$AB \perp CD$	$(x_A - x_B)(y_C - y_D) - (y_A - y_B)(x_C - x_D) = 0 \wedge (x_A - x_B)(x_A - x_B) + (y_A - y_B)(y_A - y_B) \neq 0 \wedge (x_C - x_D)(x_C - x_D) + (y_C - y_D)(y_C - y_D) \neq 0$

Formalization technique: bootstrapping

Manually bet, cong, equality, col

Automatically midpoint, right triangles, parallelism and perpendicularity

Continuity properties

- Dedekind

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- Archimedes

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- Aristotle

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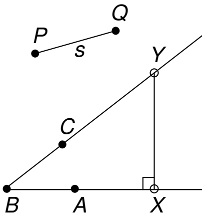
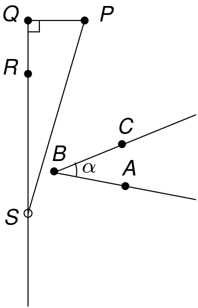


Continuity properties

- Dedekind
- Archimedes
- Aristotle
- Greenberg



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Continuity properties

- Dedekind

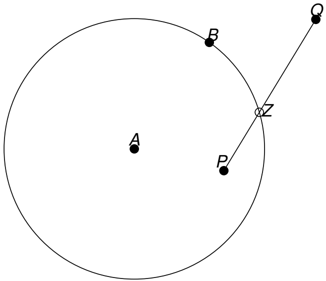
  $\Downarrow$
- Archimedes

  $\Downarrow$
- Aristotle

  $\Downarrow$
- Greenberg

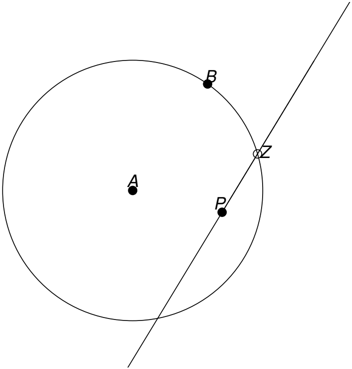
Segment-Circle / Line-Circle continuity

- Circle-Segment



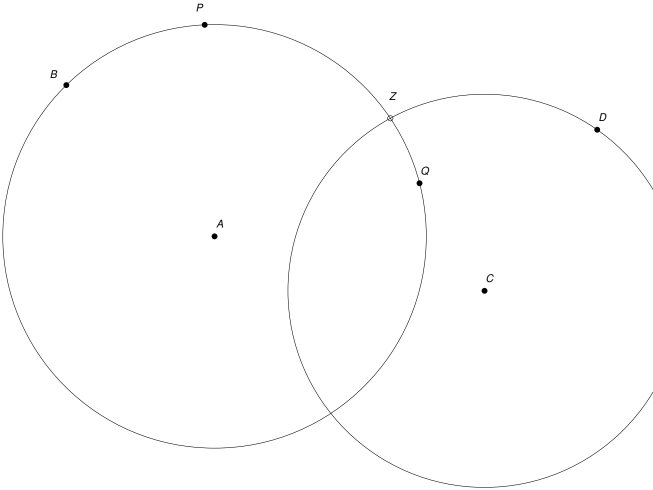
Segment-Circle / Line-Circle continuity

- Circle-Segment
- Circle-Line

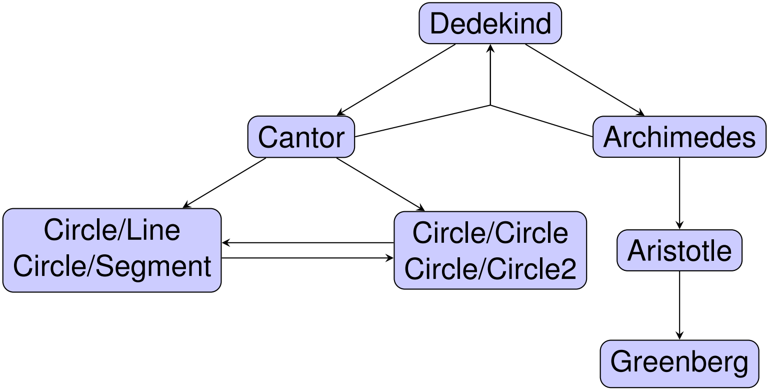


Segment-Circle / Line-Circle continuity

- Circle-Segment
- Circle-Line
- Circle-Circle



Continuity (overview)



Algebra/Geometry

Continuity	Axiom
	ordered Pythagorean field[5]
circle/line continuity	ordered Euclidean field [6]
FO Dedekind cuts	real closed field [7]
Dedekind	reals

[5] the sum of squares is a square
[6] positive are square
[7] F is euclidean and every polynomial of odd degree has at least one root in F.

Logic

Intuitionist logic [8]

- Assuming : $\forall A, B$: Points, $A = B \vee A \neq B$
- We prove : excluded middle for all other predicates,

[8] Pierre Boutry et al. (2014). "A short note about case distinctions in Tarski's geometry". In: *Automated Deduction in Geometry 2014. Proceedings of ADG 2014*

Logic

Intuitionist logic [8]

- Assuming : $\forall A, B$: Points, $A = B \vee A \neq B$
- We prove : excluded middle for all other predicates, except line intersection

[8] Pierre Boutry et al. (2014). "A short note about case distinctions in Tarski's geometry". In: *Automated Deduction in Geometry 2014. Proceedings of ADG 2014*

Constructive geometry

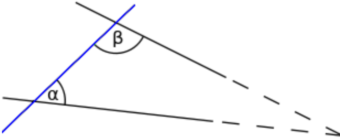
- Jan von Plato (1998). “A constructive theory of ordered affine geometry”. In: *Indagationes Mathematicae*. Vol. 9
- Michael Beeson (2015a). “A constructive version of Tarski’s geometry”. In: *Annals of Pure and Applied Logic* 166.11
- Michael Beeson (2015b). “Constructive geometry and the parallel postulate”. In: *Bulletin of Symbolic Logic* accepted pending revisions

Outline

1. Overview of GeoCoq
2. Arithmetization of Geometry
3. A mechanized study about the parallel postulates
4. Towards a formalization of the Elements

Euclid 5th postulate

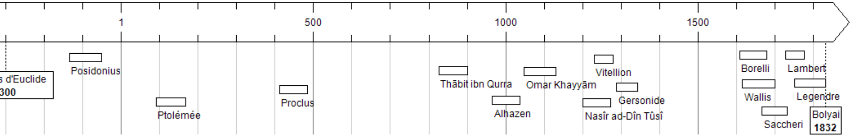
"If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough."



History

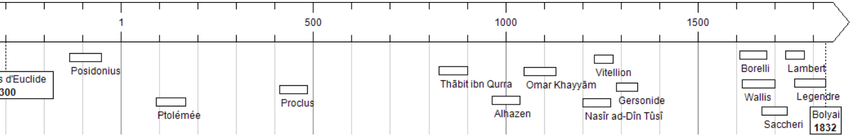
- A less obvious postulate

History

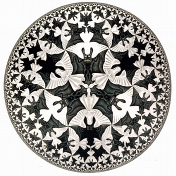


- A less obvious postulate
- Incorrect proofs during centuries

History

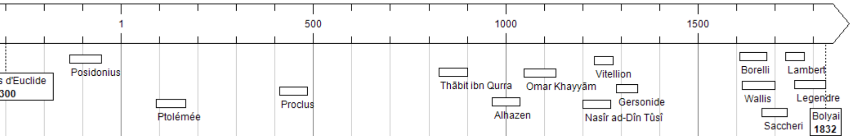


- A less obvious postulate
- Incorrect proofs during centuries
- Independence

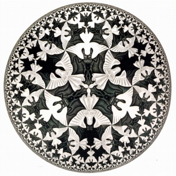



Escher, Circle Limit IV, 1960

History



- A less obvious postulate
- Incorrect proofs during centuries
- Independence
- Some equivalent statements




Escher, Circle Limit IV, 1960

A long history of incorrect proofs . . .

In 1763, Klügel [9] provides a list of 30 failed attempts at proving the parallel postulate.

Examples:

- Ptolémée uses implicitly Playfair's postulate (uniqueness of the parallel).
- Proclus uses implicitly "Given two parallel lines, if a line intersect one of them it intersects the other".
- Legendre published several incorrect proofs in its *best-seller* "Éléments de géométrie".

[9] G. S. Klugel (1763). "Conatuum praecipuorum theoriam parallelarum demonstrandi recensio". PhD thesis. Schultz, Göttingen

Mistakes

- Circular arguments

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- Implicit assumptions

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  - `parallelogram2` $ABCD$:= $AB \parallel CD \wedge AB \equiv CD \wedge$ `Convex` $ABCD$

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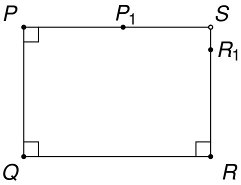
  - `parallelogram` $ABCD$ $:=$ $AB \parallel CD \wedge AD \parallel BC$
  - `parallelogram2` $ABCD$ $:=$ $AB \parallel CD \wedge AB \equiv CD \wedge$ `Convex` $ABCD$

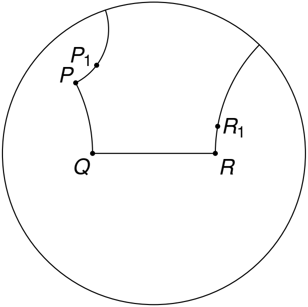
Warning !

(`parallelogram2` $ABCD \Leftrightarrow$ `parallelogram2` $BCDA$) $\Leftrightarrow$ `Euclid5`

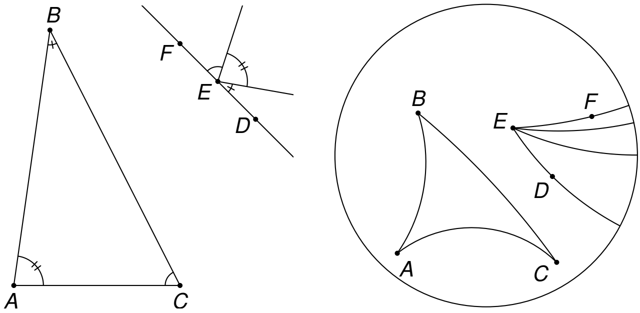
Bachmann's Lotschnittaxiom

If $p \perp q$, $q \perp r$ and $r \perp s$ then p and s meet.

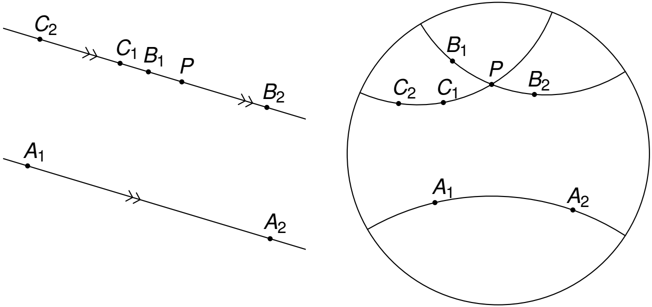




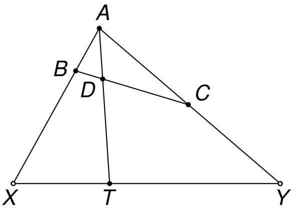
Triangle postulate

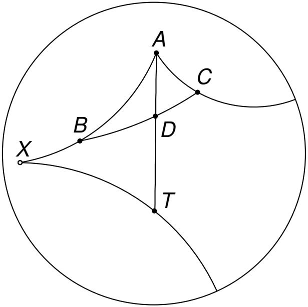


Playfair's postulate

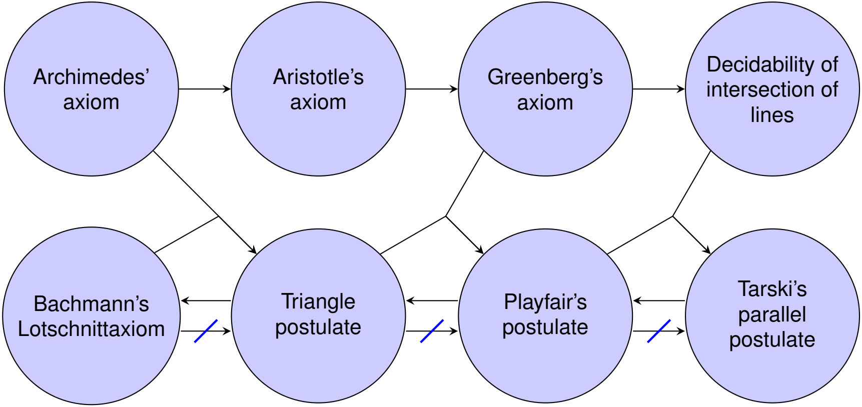


Tarski's postulate



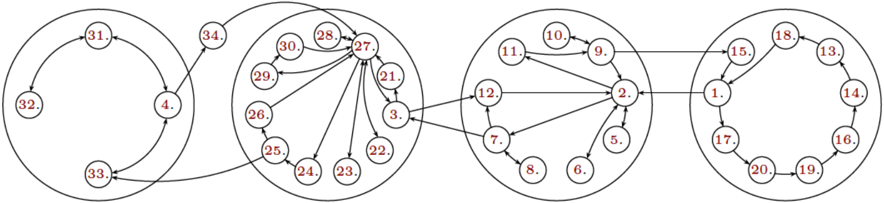


Four groups



Archimedes' axiom
Aristotle's axiom
Greenberg's axiom
Decidability of intersection of lines
Bachmann's Lotschnittaxiom
Triangle postulate
Playfair's postulate
Tarski's parallel postulate

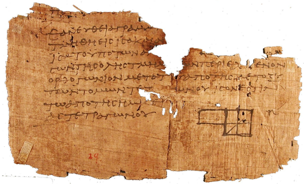
Sorting 34 postulates



Outline

The Elements

- A very influential mathematical book (more than 1000 editions).
- First known example of an axiomatic approach.



Book 2, Prop V, Papyrus d'Oxyrhynchus (year 100)

Euclid

Our project

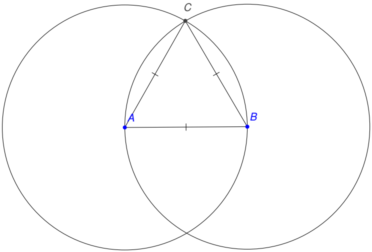
- Joint work with Charly Gries and Gabriel Braun
- Mechanizing proofs of Euclid's **statements**
- Not Euclid's proofs!
- Trying to minimize the assumptions:
  - Parallel postulate
  - Elementary continuity
  - Archimedes' axiom

Example

Proposition (Book I, Prop 1)

Let A and B be two points, build an equilateral triangle on the base AB.

Proof: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ the circles of center A and B and radius AB. Take C at the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$. The distance AB is congruent to AC, and AB is congruent to BC. Hence, ABC is an equilateral triangle.



Book I, Prop 1

We prove two statements:

1. Assuming no continuity, but the parallel postulate.
2. Assuming circle/circle continuity, but not the parallel postulate.

Pambuccian has shown that these assumptions are minimal.

```
Section Book_1_prop_1_euclidean.
Context `{TE:Tarski_2D_euclidean}.

Lemma prop_1_euclidean :
 forall A B,
  exists C, Cong A B A C /\ Cong A B B C.
Proof. ... Qed.

End Book_1_prop_1_euclidean.
```

```
Section Book_1_prop_1_circle_circle.
Context `{TE:Tarski_2D}.

Lemma prop_1_circle_circle :
circle_circle_bis ->
 forall A B,
  exists C, Cong A B A C /\ Cong A B B C.
Proof.
intros.
unfold circle_circle_bis in H.
destruct (H A B B A A B) as [C [HC1 HC2]];Circle.
exists C.
unfold OnCircle in *.
split;Cong.
Qed.

End Book_1_prop_1_circle_circle.
```

Work in progress: current status

Book I	Prop 1-34, 37, 46-47
Book II	
Book III	Prop 2-6,9-14 18

Synthetic vs Algebraic Approaches

Claim

Mixing the synthetic and algebraic approaches is useful:

- Synthetic approach for neutral geometry.
- Grobner basis for unordered Euclidean geometry.
- Proving existential by hand.

Questions ?