Philips Research Laboratory Brussels
Av. E. Van Becelaere 2, Box 8
B-1170 Brussels, Belgium

Working Document

A General Lattice Framework for Morphological Operations on Pictures

Christian Ronse

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Abstract: *In this work we present Serra's general lattice model [Serra-Th] for morphological operations on pictures (dilations, erosions, openings and closings), and introduce a new class of openings containing as particular case the narrow peak erasers introduced in [Ronse-conf].*

Keywords: supremum, infimum, complete lattice, dilation, erosion, opening, closing, duality.

I. Introduction

Linear methods have been successfully applied in the processing of acoustical signals. They have then been generalized to image processing. However it becomes widely felt that they are unsufficient for this purpose. One reason is that there is a fundamental difference between acoustical signals and pictorial ones [Serra-ICPR]: while the former obey to the law of superposition (two sound waves emitted at the same time combine linearly), the latter are 2-D projections of 3-D objects and they obey to the law of occlusion (one object in front of another hides it). Another reason is the need to relate grey-level pictures to their binary counterparts, whose properties rest on set-theoretic operations on the subsets of the plane.

More evidence on the unsufficiency of linear, Fourier-type, techniques comes from the study of human vision. For example there exist certain types of textures whose spatial organization cannot be recovered by Fourier techniques, but well by grouping processes (see Figure 2–2 of [Marr]). On the other hand, there are pairs of textures which differ totally in their Fourier spectrum, but which are difficult to discriminate by the human eye (see in particular [Mayhew-Frisby]). Moreover, two similar two-tone structures of different sizes can produce different spatial effects; this is for example the case with chessboard patterns, in which the visual proeminence of the diagonal directions (in conformity with the diagonal structure of the Fourier spectrum) gradually disappears when the squares become larger (see Figure 2–24 of [Marr]).

Hence increasing attention is given to non-linear methods for the processing of pictorial signals. Mathematical morphology is one paticular discipline in non-linear image processing [Serra-ICPR] which relies on set-theoretical aspects of pictures. The concepts and tools that it has produced have been applied to various practical problems [Serra-IAMM].

Binary image transformations based on set-theoretical operations date from Minkowski, and have been studied by many authors. A systematic treatment of this topic is mainly due to Matheron [Matheron-RSIG] and Serra [Serra-IAMM]. Their generalization to grey-level images [Goetcherian,Meyer,Sternberg] has been formalized in Chapter XII of [Serra-IAMM]. It is no wonder that such operations rely on the *supremum* and *infimum* of a collection of grey-level images as generalizations of the union and intersection of a collection of subsets of the space.

However, these morphological operations for grey-level pictures were considered in a restrictive framework; for example, the image space and the set of grey-levels must be unbounded; these two requirement cannot be met in computer-based applications, where we must work with finite spaces and bounded discrete grey-levels. As we will see in some examples later, the extension of morphological operations and of their properties to the case where the set of grey-levels is bounded requires some precautions. It is thus interesting to analyse in a general framework, wider than binary or grey-level images on the Euclidean space, transformations based on the supremum and infimum. The corresponding structure is the *complete lattice* [Birkhoff,Dubreil]. The interest of such an abstract formalism is twofold:

first, to prove in a concise way general results applicable to various image structures; second to distinguish between the properties of images which rely on the structure of complete lattice and those which depend upon the arithmetic properties of the set of grey-levels or the geometric properties of the Euclidean or digital space.

In Section II we give the mathematical prerequisites for our work: the definition and basic properties of *complete lattices* and related structures.

In Sections III, IV, and V we introduce the basic operations on complete lattices: dilations, erosions, openings, and closings, and give their main properties. Most of the material presented there comes from [Serra-Th] and [Matheron-RSIG].

In Section VI we show how to build openings and closings by combining erosions and dilations. The first method, generalizing the "morphological opening by a structuring element" of [Serra-IAMM], comes from [Serra-Th]. The second one is an extension of the first one, and contains as particular case the "narrow peak erasers" defined in [Ronse-conf].

Several examples relate the concepts and operations described in this work to known morphological operations used in image processing. The usefulness of our general approach is discussed in Section VII, where we describe also some possible practical applications.

II. Complete lattices and duality

In this section we will recall some basic definitions and properties of ordered sets and complete lattices. They come from [Birkhoff] and [Dubreil]. Our definitions and notation given here are a compromise between these two sources.

Consider a set $\mathcal{S}$; a binary relation $\preceq$ on $\mathcal{S}$ is called an *order relation* if it is

(i) *reflexive*: for any $X \in \mathcal{S}$, $X \preceq X$;

(ii) *antisymmetric*: for any $X, Y \in \mathcal{S}$, if $X \preceq Y$ and $Y \preceq X$, then $X = Y$;

(i) *transitive*: for any $X, Y, Z \in \mathcal{S}$, if $X \preceq Y$ and $Y \preceq Z$, then $X \preceq Z$.

We say then that $\mathcal{S}$ is an *ordered set*. The expression "$X \preceq Y$" is read as "X is below Y". The reverse relation $\succeq$ (defined by $X \succeq Y$ iff $Y \preceq X$) is also an order relation; the expression "$X \succeq Y$" is then read as "X is above Y". There are many examples of ordered sets: the set of parts of a set, ordered by inclusion; the set of real numbers ordered by $\leq$; the set of natural integers with the relation "divides".

The order relation $\preceq$ gives rise to the *strict order relation* $\prec$ defined by $X \prec Y$ iff $X \preceq Y$ and $X \neq Y$.

Given $X \in \mathcal{S}$ and $\mathcal{T} \subseteq \mathcal{S}$, we say that X is an *upper bound* of $\mathcal{T}$ if for any $Y \in \mathcal{T}$ we have $X \succeq Y$, and that X is a *lower bound* of $\mathcal{T}$ if for any $Y \in \mathcal{T}$ we have $X \preceq Y$. Note that when $\mathcal{T}$ is empty, any element of $\mathcal{S}$ is both an upper and a lower bound of $\mathcal{T}$.

A *supremum* of $\mathcal{T}$ in $\mathcal{S}, \preceq$ is a lowest upper bound of $\mathcal{T}$, in other words an upper bound of $\mathcal{T}$ which is below any other upper bound of $\mathcal{S}$. *In extenso*, it is an element X of

$\mathcal{S}$ such that $X \succeq Y$ for any $Y \in \mathcal{T}$, and for any $Z \in \mathcal{S}$ such that $Z \succeq Y$ for any $Y \in \mathcal{T}$, we have $X \preceq Z$. *If it exists*, it is necessarily unique (by the antisymmetry of $\preceq$).

Conversely, an *infimum* of $\mathcal{T}$ in $\mathcal{S}, \preceq$ is a highest lower bound of $\mathcal{T}$, in other words a lower bound of $\mathcal{T}$ which is above any other lower bound of $\mathcal{S}$. *If it exists*, it is unique.

Let us give some examples in set theory: (1°) If $\mathcal{S}$ is the set of parts of a set E, the supremum and infimum in $\mathcal{S}, \subseteq$ are the union and the intersection respectively. (2°) If E is finite and $\mathcal{S}'$ is the set of all subsets of E of even size, then the supremum and infimum in $\mathcal{S}', \subseteq$ of a subset $\mathcal{T}$ of $\mathcal{S}'$ are not always defined, because the union or intersection of even-sized sets may have an odd size. (3°) If E is a vector space and if $\mathcal{S}''$ is the set of all vector subspaces of E, then the supremum in $\mathcal{S}'', \subseteq$ of a subset $\mathcal{T}$ of $\mathcal{S}''$ is the sum of all vector subspaces which are elements of $\mathcal{T}$.

These examples show that the existence and the value of the supremum and infimum of $\mathcal{T}$ depend on the set $\mathcal{S}$ (or the subset of it) in which they are taken. The supremum and infimum of $\mathcal{T}$ in $\mathcal{S}, \preceq$ will be written

$$\sup_{\mathcal{S},\preceq} \mathcal{T} \qquad \text{and} \qquad \inf_{\mathcal{S},\preceq} \mathcal{T}$$

respectively, provided that they exist. However when there is no ambiguity on the ordered set $\mathcal{S}, \subseteq$ in which the supremum and infimum are taken, we will simply write $\sup \mathcal{T}$ or $\bigvee \mathcal{T}$ for $\sup_{\mathcal{S},\preceq} \mathcal{T}$, and $\inf \mathcal{T}$ or $\bigwedge \mathcal{T}$ for $\inf_{\mathcal{S},\preceq} \mathcal{T}$.

Two elements of the ordered set $\mathcal{S}$ are important, when they exist: the *universal bounds* [Birkhoff]. They are the highest element $\mathbf{I}$ and the lowest element $\mathbf{O}$. In [Dubreil] they are called the *universal* element and the *null* element respectively. Clearly

$$\mathbf{I} = \sup_{\mathcal{S},\preceq} \mathcal{S} \qquad \text{and} \qquad \mathbf{O} = \inf_{\mathcal{S},\preceq} \mathcal{S}. \tag{1}$$

As we said above, any element of $\mathcal{S}$ is both an upper and a lower bound of the empty subset $\emptyset$ of $\mathcal{S}$. Thus

$$\mathbf{O} = \sup_{\mathcal{S},\preceq} \emptyset \qquad \text{and} \qquad \mathbf{I} = \inf_{\mathcal{S},\preceq} \emptyset. \tag{2}$$

The two equalities in (2) correspond to the convention used in algebra which sets an empty sum equal to 0 and an empty product equal to 1.

Now we will say that $\mathcal{S}$ is a *complete lattice* if every *nonvoid* subset of $\mathcal{S}$ has a supremum and an infimum in $\mathcal{S}, \preceq$. This implies in particular that $\mathcal{S}$ contains universal bounds $\mathbf{I} = \sup \mathcal{S}$ and $\mathbf{O} = \inf \mathcal{S}$.

Note that in a complete lattice the empty set $\emptyset$ has also a supremum and an infimum, since $\mathbf{O} = \sup \emptyset$ and $\mathbf{I} = \inf \emptyset$. Hence any subset of $\mathcal{S}$ has a supremum and an infimum, not only nonvoid ones.

As said above, one usually writes $\bigvee$ and $\bigwedge$ for $\sup_{\mathcal{S},\preceq}$ and $\inf_{\mathcal{S},\preceq}$. There are some other usual conventions for notation. If a subset $\mathcal{T}$ of $\mathcal{S}$ can be written under the form

$$\{\text{expression} \mid \text{condition}\},$$

then $\bigvee \mathcal{T}$ and $\bigwedge \mathcal{T}$ can be written

$$\bigvee_{\text{condition}} \text{expression} \qquad \text{and} \qquad \bigwedge_{\text{condition}} \text{expression}$$

respectively. When $\mathcal{T}$ is finite and we have $\mathcal{T} = \{X_1, \ldots, X_n\}$, we will write

$$X_1 \vee \cdots \vee X_n \qquad \text{and} \qquad X_1 \wedge \cdots \wedge X_n$$

respectively. These two expressions use the binary operations $\vee$ and $\wedge$ (the supremum and infimum of two elements of $\mathcal{S}$) which are idempotent, commutative, and associative (this means that for $X, Y, Z \in \mathcal{S}$, $X \vee X = X$, $X \vee Y = Y \vee X$, $X \vee (Y \vee Z) = (X \vee Y) \vee Z$, and similarly for $\wedge$).

We said above that the reverse $\succeq$ of an order relation $\preceq$ is itself an order relation. This reversion extends then to the supremum and infimum, since we have for any $\mathcal{T} \subseteq \mathcal{S}$:

$$\begin{aligned} \sup_{\mathcal{S},\succeq} \mathcal{T} &= \inf_{\mathcal{S},\preceq} \mathcal{T}; \\ \inf_{\mathcal{S},\succeq} \mathcal{T} &= \sup_{\mathcal{S},\preceq} \mathcal{T}. \end{aligned} \tag{3}$$

The universal bounds of $\mathcal{S}, \succeq$ are those of $\mathcal{S}, \preceq$, but reversed.

Thus if $\mathcal{S}, \preceq$ is a complete lattice, with supremum $\bigvee$, infimum $\bigwedge$, null element $\mathbf{O}$, and universal element $\mathbf{I}$, then $\mathcal{S}, \succeq$ is also a complete lattice, but this time with supremum $\bigwedge$, infimum $\bigvee$, null element $\mathbf{I}$, and universal element $\mathbf{O}$. We call it the *dual lattice* of $\mathcal{S}$.

This implies that any general property of a lattice is true for its dual lattice. In other words, in every general statement on lattices, we can reverse $\preceq$ and $\succeq$, $\bigvee$ and $\bigwedge$, $\mathbf{O}$ and $\mathbf{I}$. This important fact is called the *duality principle*.

Given a complete lattice $\mathcal{S}, \preceq$, a subset $\mathcal{M}$ of $\mathcal{S}$ is called a *Moore family* [Dubreil] if $\mathbf{I} \in \mathcal{M}$ and $\bigwedge \mathcal{U} \in \mathcal{M}$ for any nonvoid subset $\mathcal{U}$ of $\mathcal{M}$. Recalling that $\bigwedge \emptyset = \mathbf{I}$ (see (2)), $\mathcal{M}$ is a Moore family iff $\bigwedge \mathcal{U} \in \mathcal{M}$ for any subset $\mathcal{U}$ of $\mathcal{M}$.

An interesting fact is that a Moore family forms itself a complete lattice, but with a different supremum than $\bigvee$. Indeed, for any $\mathcal{U} \subseteq \mathcal{M}$, the set

$$\mathrm{UB}(\mathcal{U}, \mathcal{M}) = \{Z \in \mathcal{M} \mid \forall X \in \mathcal{U}, X \preceq Z\}$$

of upper bounds of $\mathcal{U}$ in $\mathcal{M}$ is not empty, since it contains $\mathbf{I}$. Thus its infimum $\bigwedge \mathrm{UB}(\mathcal{U}, \mathcal{M})$ belongs to $\mathcal{M}$. Now for every $X \in \mathcal{U}$, X is a lower bound of $\mathrm{UB}(\mathcal{U}, \mathcal{M})$; hence (by definition of the infimum), $X \preceq \bigwedge \mathrm{UB}(\mathcal{U}, \mathcal{M})$. This means that $\bigwedge \mathrm{UB}(\mathcal{U}, \mathcal{M})$ is an upper bound of $\mathcal{U}$ in $\mathcal{M}$:

$$\bigwedge \mathrm{UB}(\mathcal{U}, \mathcal{M}) \in \mathrm{UB}(\mathcal{U}, \mathcal{M}).$$

Now it is clear that $\bigwedge \mathrm{UB}(\mathcal{U}, \mathcal{M})$ is below any other element of $\mathrm{UB}(\mathcal{U}, \mathcal{M})$. It is thus the lowest element of the set $\mathrm{UB}(\mathcal{U}, \mathcal{M})$ of upper bounds of $\mathcal{U}$ in $\mathcal{M}$, in other words it is the supremum of $\mathcal{U}$ in $\mathcal{M}$:

$$\bigwedge \mathrm{UB}(\mathcal{U}, \mathcal{M}) = \sup_{\mathcal{M}} \mathcal{U}.$$

Hence $\mathcal{M}$ is a complete lattice, with the same infimum $\bigwedge$ as in $\mathcal{S}$, but not necessarily the same supremum. Note also that $\mathcal{M}$ has the same universal element $\mathbf{I}$ as $\mathcal{S}$, but not necessarily the same null element.

Consider for example a vector space V, and let $\mathcal{S}$ be the set of parts of V. This set, ordered by inclusion, is a complete lattice, with the union as supremum, the intersection as infimum, and V and $\emptyset$ as universal bounds. Now let $\mathcal{M}$ be the set of vector subspaces of V. Then $\mathcal{M}$ is a Moore family in $\mathcal{S}$. Indeed, V is a vector subspace of V, and the intersection of vector subspaces of V is itself a vector subspace of V. Then $\mathcal{M}$ is a complete lattice, with again the intersection as infimum, but with a supremum which is not the union, because the union of vector subspaces of V is not a vector subspace V. In fact the supremum in $\mathcal{M}$ of a subset $\mathcal{U}$ of $\mathcal{M}$ is the *sum* of all vector spaces elements of $\mathcal{U}$. Moreover, $\mathcal{M}$ has the same universal element V as $\mathcal{S}$, but not the same null element: the null element of $\mathcal{S}$ is $\emptyset$, while the null element of $\mathcal{M}$ is the zero vector space.

A subset $\mathcal{N}$ of a complete lattice $\mathcal{S}, \preceq$ will be called by us a *dual Moore family* if it is a Moore family of the dual lattice $\mathcal{S}, \succeq$, in other words if $\mathbf{O} \in \mathcal{N}$ and $\bigvee \mathcal{U} \in \mathcal{N}$ for any nonvoid subset $\mathcal{U}$ of $\mathcal{N}$. Recalling that $\bigvee \emptyset = \mathbf{O}$ (see (2)), $\mathcal{N}$ is a dual Moore family iff $\bigvee \mathcal{U} \in \mathcal{N}$ for any subset $\mathcal{U}$ of $\mathcal{N}$. Again a dual Moore family is a complete lattice.

We have given here the basic concepts and properties concerning the general framework in which we will define morphological operations: ordered sets, supremum and infimum, universal bounds, complete lattices, duality, and Moore families. Nothing more is needed.

III. Picture operators, dilations, and erosions

We take a complete lattice $\mathcal{L}$ with the order relation $\preceq$, supremum $\bigvee$, infimum $\bigwedge$, null element $\mathbf{O}$ and universal element $\mathbf{I}$. Elements of $\mathcal{L}$ will be called *pictures* and written as capital letters X, Y, Z, etc..

In practice $\mathcal{L}$ will correspond to a particular set of pictures we work with. For example, if we consider binary images on a Euclidean or digital space E, $\mathcal{L}$ will be the set of parts of E, ordered by inclusion, with the union and intersection as supremum and infimum.

On the other hand, if we consider grey-level images on E, $\mathcal{L}$ will be the set of maps $X : E \to D : p \mapsto X(p)$, where D is the set of grey-levels; D will be a *closed* subset of

$\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$, so that for any subset T of D, $\inf_{D,\leq} T$ and $\sup_{D,\leq} T$ will be defined. The order relation $\preceq$ on $\mathcal{L}$ will be defined by

$$X \preceq Y \qquad \text{iff} \qquad \forall p \in E, \quad X(p) \leq Y(p)$$

for $X, Y \in \mathcal{L}$. Then $\bigvee$ and $\bigwedge$ are defined by setting

$$\begin{aligned} & [\bigvee \mathcal{T}](p) = \sup_{D,\leq}\{X(p) \mid X \in \mathcal{T}\} \\ \text{and} \qquad & [\bigwedge \mathcal{T}](p) = \inf_{D,\leq}\{X(p) \mid X \in \mathcal{T}\} \end{aligned}$$

for any $p \in E$ and $\mathcal{T} \subseteq \mathcal{L}$.

We will consider the set $\mathcal{O}$ of all transformations on $\mathcal{L}$, in other words the set $\mathcal{L}^{\mathcal{L}}$ of maps $\mathcal{L} \to \mathcal{L}$. Given $X \in \mathcal{L}$ and $\theta \in \mathcal{O}$, θ maps X to $\theta(X)$, which will be called the *transform of X by θ*. Elements of $\mathcal{O}$ will be called *operators*. Several particular operators must be mentionned right now:

— the identity 1 defined by $1(X) = X$ for every $X \in \mathcal{L}$;

— the constant operators $\langle Z \rangle$, where $Z \in \mathcal{L}$, defined by $\langle Z \rangle(X) = Z$ for every $X \in \mathcal{L}$. (Note that Z can be any expression, not only a simple symbol.)

Other operators will be written by lowercase greek letters β, γ, etc., with the letters $\alpha, \delta, \varepsilon, \varphi$ being reserved to openings, dilations, erosions, and closings.

The *composition* $\eta\theta$ of the operator θ by the operator η is defined by setting

$$\eta\theta(X) = \eta\big(\theta(X)\big) \tag{4}$$

for every picture X. This operation is associative, in other words $\beta[\eta\theta] = [\beta\eta]\theta$ for any $\beta, \eta, \theta \in \mathcal{O}$. As shown in this example, we will use square brackets [] instead of parantheses () for grouping in expressions built with operators, in order to avoid confusion with the transform of a picture by an operator. Thus for example $\beta[1 \vee \langle Y \rangle]$ will be the composition of $1 \vee \langle Y \rangle$ by β, while $\beta[X \vee Y]$ will be the transform of $X \vee Y$ by β.

An interesting fact is that the structure of complete lattice of $\mathcal{L}$ extends to $\mathcal{O}$ in the same way as we extended this structure from the set D of grey-levels to the set of grey-level images $E \to D$ in the second example given at the beginning of this section. First the order relation $\preceq$ on $\mathcal{L}$ can be transposed to an order relation on $\mathcal{O}$ by setting for $\eta, \theta \in \mathcal{O}$:

$$\eta \preceq \theta \qquad \text{iff} \qquad \forall X \in \mathcal{L}, \quad \eta(X) \preceq \theta(X). \tag{5}$$

We can now define the operations $\bigvee$ and $\bigwedge$ on $\mathcal{O}$ by setting for any $X \in \mathcal{L}$ and $\mathcal{Q} \subseteq \mathcal{O}$:

$$\begin{aligned} & [\bigvee \mathcal{Q}](X) = \bigvee_{\eta \in \mathcal{Q}} \eta(X) \\ \text{and} \qquad & [\bigwedge \mathcal{Q}](X) = \bigwedge_{\eta \in \mathcal{Q}} \eta(X). \end{aligned} \tag{6}$$

Then it is easy to see that $\mathcal{O}, \preceq, \bigvee, \bigwedge$ is a complete lattice with null element $\langle \mathbf{O} \rangle$ and universal element $\langle \mathbf{I} \rangle$. Note that $\emptyset$ is a subset of both $\mathcal{L}$ and $\mathcal{O}$, which leads to the following ambiguity (see (2)):

$$\bigvee \emptyset = \mathbf{O} \quad \text{and} \quad \bigwedge \emptyset = \mathbf{I} \qquad \text{in } \mathcal{L};$$
$$\bigvee \emptyset = \langle \mathbf{O} \rangle \quad \text{and} \quad \bigwedge \emptyset = \langle \mathbf{I} \rangle \qquad \text{in } \mathcal{O}.$$

When it arises, this ambiguity will be removed by the context.

We said in the Introduction that morphological operators were based on the supremum and the infimum. We can introduce now such operators:

Definition 1. Let $\beta \in \mathcal{O}$. Then we say that:

(a) β is *increasing* if for every $X, Y \in \mathcal{L}$, $X \preceq Y$ implies that $\beta(X) \preceq \beta(Y)$.

(b) [Serra-Th] β is a *dilation* if for every $\mathcal{T} \subseteq \mathcal{L}$, $\beta(\bigvee \mathcal{T}) = \bigvee_{X \in \mathcal{T}} \beta(X)$.

(c) [Serra-Th] β is an *erosion* if for every $\mathcal{T} \subseteq \mathcal{L}$, $\beta(\bigwedge \mathcal{T}) = \bigwedge_{X \in \mathcal{T}} \beta(X)$.

Note that in (b) and (c) we must also take into account the case where $\mathcal{T}$ is empty. Thus (by (2)), a dilation preserves $\mathbf{O}$ and an erosion preserves $\mathbf{I}$. Thanks to (6) and the existence of constant operators, it is easily seen that for any $\beta \in \mathcal{O}$ we have:

(a') β is increasing iff for any $\eta, \theta \in \mathcal{O}$, $\eta \preceq \theta$ implies that $\beta\eta \preceq \beta\theta$.

(b') β is a *dilation* iff for every $\mathcal{Q} \subseteq \mathcal{O}$, $\beta[\bigvee \mathcal{Q}] = \bigvee_{\gamma \in \mathcal{Q}} \beta\gamma$.

(c') β is an *erosion* iff for every $\mathcal{Q} \subseteq \mathcal{O}$, $\beta[\bigwedge \mathcal{Q}] = \bigwedge_{\gamma \in \mathcal{Q}} \beta\gamma$.

Increasingness plays a fundamental role in mathematical morphology (see [Serra-IAMM,Serra-Th]). Note that this concept is auto-dual from the point of view of complete lattices. Therefore to every statement on increasing operators corresponds a dual statement, where we reverse $\preceq$ and $\succeq$, $\bigvee$ and $\bigwedge$, $\mathbf{O}$ and $\mathbf{I}$, etc.. Let us write $\mathcal{I}$ for the set of increasing operators. We have the following fundamental result:

Proposition 1. (i) *$\mathcal{I}$ is closed under composition and contains* 1.

(ii) *$\mathcal{I}$ is a complete lattice with universal bounds $\langle \mathbf{O} \rangle$ and $\langle \mathbf{I} \rangle$.*

Proof. Take $X, Y \in \mathcal{L}$ such that $X \preceq Y$.

(i) It is obvious that $1 \in \mathcal{I}$.

Let $\eta, \theta \in \mathcal{I}$. As θ is increasing, we get $\theta(X) \preceq \theta(Y)$, and as η is increasing, we obtain $\eta(\theta(X)) \preceq \eta(\theta(Y))$. Thus $\eta\theta \in \mathcal{I}$.

(ii) It is obvious that $\langle \mathbf{O} \rangle, \langle \mathbf{I} \rangle \in \mathcal{I}$.

Consider now a non-empty subset $\mathcal{Q}$ of $\mathcal{I}$. Let us show that $\beta = \bigvee \mathcal{Q} \in \mathcal{I}$. For every $\eta \in \mathcal{Q}$, we have $\eta(X) \preceq \eta(Y)$, since η is increasing. Now $\eta \preceq \beta$ (since η intervenes in the

$\bigvee$-decomposition of β), and so $\eta(Y) \preceq \beta(Y)$. Thus $\eta(X) \preceq \beta(Y)$. Hence

$$\beta(X) = \bigvee_{\eta \in \mathcal{Q}} \eta(X) \preceq \beta(Y),$$

and so β is increasing.

By duality, we have also $\bigwedge \mathcal{Q} \in \mathcal{I}$. Hence $\mathcal{I}$ is a complete lattice. ∎

Let us now turn to dilations and erosions. As we will see in the rest of this paper, they are the basic tools for many morphological operators. We will write $\mathcal{D}$ for the set of dilations and $\mathcal{E}$ for the set of erosions. Dilations will be written $\delta, \delta', \delta_1$, etc., while erosions will be written $\varepsilon, \varepsilon', \varepsilon_1$, etc..

It is clear that erosions and dilations are dual concepts from the complete lattice point of view (see Section II). Hence to every statement on dilations corresponds a dual statement on erosions, and vice versa. Similarly to every example of dilation corresponds a dual example of erosion.

Let us illustrate these two operations. Consider first the case of binary pictures on E, in other words subsets of E. Given a dilation δ, for any $X \subseteq E$ we have

$$\delta(X) = \bigcup_{x \in X} \delta(\{x\}), \tag{7}$$

since $X = \bigcup_{x \in X} \{x\}$. In practice, $\delta(\{x\})$ will usually be a window $W(x)$ containing x. For example, if E is a digital grid, $W(x)$ can be the set of all points of E at distance at most k from x, and then $\delta(X)$ will be the set of all points of E at distance at most k from X. The effect of applying such a dilation on X is to "expand" it. One often assumes that the windows $W(x)$ are uniform; if we consider the Euclidean space as a vector space, this means that there is a set B (called a *structuring element*), such that each $W(x)$ is the set

$$B_x = \{x + b \mid b \in B\}. \tag{8}$$

Then $\delta(X)$ will be equal to

$$X \oplus B = \bigcup_{x \in X} B_x = \{x + b \mid x \in X, b \in B\}. \tag{9}$$

The operation $\oplus$ is called the *Minkowski addition*.

Let us now turn to erosions. Write $\mathbf{C}(X)$ for the complement of a set X in E. Given an erosion ε, for any $X \subseteq E$ we have

$$\varepsilon(X) = \bigcap_{y \in \mathbf{C}(X)} \varepsilon(\mathbf{C}(\{y\})) \tag{10}$$

since $X = \bigcap_{y \in \mathbf{C}(X)} \mathbf{C}(\{y\})$. Applying it to $\mathbf{C}(X)$ and complementing the whole, (10) becomes

$$\mathbf{C}\big(\varepsilon(\mathbf{C}(X))\big) = \bigcup_{y \in X} \mathbf{C}\big(\varepsilon\big(\mathbf{C}(\{y\})\big)\big).$$

Comparing this with (7), it is easy to see that applying an erosion on a subset of E amounts to applying a dilation on its complement, since the set-theoretic operation of complementation is an isomorphism between the lattice E and its dual. In practice we will usually have $\varepsilon\big(\mathbf{C}(\{x\})\big) = \mathbf{C}(W(x))$ for a window $W(x)$ containing x. The effect of applying such an erosion on X is to "shrink" it. Now if each $\varepsilon\big(\mathbf{C}(\{x\})\big)$ is of the form $\mathbf{C}(B_x)$, then $\varepsilon(X)$ will be equal to

$$X \ominus B = \mathbf{C}(\mathbf{C}(X) \oplus B) = \{y \in E \mid \forall b \in B, y - b \in X\}. \tag{11}$$

The operation $\ominus$ is called the *Minkowski subtraction*.

Consider next grey-level pictures $E \to \overline{\mathbf{R}}$, where $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$. Generalizations of the Minkowski addition and subtraction have been defined [Meyer,Sternberg] by taking a *structuring function* instead of a structuring element (see also Chapter XII of [Serra-IAMM]). The idea is to take a subset B of E (as in (9) and (11)), and to associate to each point of B a finite grey-level.

We call a *partial picture* a map $G : S \to \overline{\mathbf{R}}$, where $S \subseteq E$; moreover S will be called the *support* of G, and we will write $S = S(G)$. If the partial picture G is a map $S(G) \to \mathbf{R}$, then we will call it a *structuring function*.

A partial picture H (in other words a map $S(H) \to \overline{\mathbf{R}}$) can be extended to a whole picture $E \to \overline{\mathbf{R}}$ thanks to a grey-level $w \in \overline{\mathbf{R}}$; indeed, we define the *extension* $Ext_w(H)$ *of* H *by* w as the picture given by

$$Ext_w(H)(z) = \begin{cases} H(z) & \text{if } z \in S(H), \\ w & \text{otherwise,} \end{cases} \qquad \text{for} \quad z \in E. \tag{12}$$

Given a partial picture H and a point $x \in E$, the *translate of* H *by* x is the picture $H_x : S(H)_x \to \overline{\mathbf{R}}$ defined by

$$H_x(z) = H(z - x) \qquad \text{for} \quad z \in S(H)_x. \tag{13}$$

Given a structuring function $G : S(G) \to \mathbf{R}$ and $u \in \overline{\mathbf{R}}$, we define the *shifted* partial picture $\sigma_u(G) : S(G) \to \overline{\mathbf{R}}$ by setting

$$\sigma_u(G)(z) = G(z) + u \qquad \text{for} \quad z \in E. \tag{14}$$

(Note that u may be infinite). For any $B \subseteq E$, let us set

$$\widetilde{B} = \{-b \mid b \in B\}. \tag{15}$$

We can now define the dilation of a picture F by a structuring function G. Here each individual grey-level value $F(x)$ in the picture F gives rise to the whole partial picture $\sigma_{F(x)}(G_x)$, in other words a picture with grey-level $F(x)$ on x and grey-level $-\infty$ on every other point y is transformed into the whole picture $Ext_{-\infty}(\sigma_{F(x)}(G_x))$. So we define $\delta_G(F)$ by setting

$$\delta_G(F) = \bigvee_{x\in E} Ext_{-\infty}\big(\sigma_{F(x)}(G_x)\big). \tag{16}$$

For any $z \in E$, we have then

$$\delta_G(F)(z) = \sup_{x\in E} Ext_{-\infty}\big(\sigma_{F(x)}(G_x)\big)(z) = \sup_{z\in S(G)_x} \sigma_{F(x)}(G_x)(z) = \sup_{z\in S(G)_x} \big(F(x)+G(z-x)\big),$$

since $Ext_{-\infty}(\sigma_{F(x)}(G_x))(z) = -\infty$ if $z \notin S(G_x) = S(G)_x$. Note that for $z, x \in E$, $z \in S(G)_x$ iff $z - x \in S(G)$, iff $x - z \in \widetilde{S(G)}$, iff $x \in \widetilde{S(G)}_z$. Therefore we get

$$\delta_G(F)(z) = \sup_{x\in \widetilde{S(G)}_z} \big(F(x) + G(z-x)\big) \qquad \text{for} \quad z \in E. \tag{17}$$

It is relatively easy to show that δ_G is a dilation. Note the resemblance of (17) with a convolution: here we have a supremum instead of an integral, and a sum instead of a product.

The grey-level inversion takes in $\overline{\mathbf{R}}$ the role of the complementation for binary pictures. Thus applying an erosion on a picture amounts to applying a dilation to the inverted picture. Hence the erosion of a picture F by a structuring function G is defined by

$$\varepsilon_G(F) = -\delta_G(-F) = -\bigvee_{x\in E} Ext_{-\infty}\big(\sigma_{-F(x)}(G_x)\big) = \bigwedge_{x\in E} Ext_{+\infty}\big(\sigma_{F(x)}(-G_x)\big). \tag{18}$$

An expansion similar to (17) gives then:

$$\varepsilon_G(F)(z) = \inf_{x\in \widetilde{S(G)}_z} \big(F(x) - G(z-x)\big) \qquad \text{for} \quad z \in E, \tag{19}$$

We can show without much pain that ε_G is an erosion.

When $S(G)$ is finite, the supremum and infimum in (17) and (19) reduce to a maximum and a minimum respectively. When $S(G)$ is empty, we have a void supremum and infimum, and so δ_G and ε_G reduce to the null and universal operators $\langle\mathbf{O}\rangle$ and $\langle\mathbf{I}\rangle$ respectively; we call then G the empty structuring function.

Let us give a visual interpretation of δ_G and ε_G in the case where $E = \mathbf{R}$. Consider a picture $F : \mathbf{R} \to \overline{\mathbf{R}}$ and the structuring function $G : S(G) \to \mathbf{R}$. For every $x \in R$, we shift G horizontally by x and vertically by $F(x)$, and obtain a new partial picture $\sigma_{F(x)}(G_x)$; then the supremum of these partial pictures (visually, the upper enveloppe of the family of curves representing them) is $\delta_G(F)$. We do the same thing with $-G$ for every $x \in R$; then the infimum of these partial pictures (visually, the lower enveloppe of the family of curves representing them) is ε_G. We illustrate these two constructions in Figure 1 for a structuring function G shaped as a "pencil tip".

In Chapter XII of [Serra-IAMM], one shows how formulas (17) and (19) can be related to the Minkowski addition and subtraction thanks to the *umbra*. Given a picture F, the *umbra* $U(F)$ of F is the set of all ordered pairs $(x, g) \in E \times \mathbf{R}$ such that $F(x) \geq g$. Given a structuring picture G, we can identify it with the set of ordered pairs $(x, G(x))$ for $x \in S(G)$. Then it is not too hard to see [Serra-IAMM] that in $E \times \mathbf{R}$ we have

$$U(\delta_G(F)) = U(F) \oplus G = U(F) \oplus U(G)$$

and

$$U(\varepsilon_G(F)) = U(F) \ominus (-G),$$

where $-G$ is the set of $(x, -G(x))$ for $x \in S(G)$.

A well-known particular case of dilation and erosion is given by the structuring function G defined by:

$$G(z) = 0 \qquad \text{for} \quad z \in S(G). \tag{20}$$

We call it a *flat* structuring function. Here (17) and (19) reduce to

$$\begin{aligned} \delta_G(F)(z) &= \sup_{x \in \widetilde{S(G)}_z} F(x), \\ \varepsilon_G(F)(z) &= \inf_{x \in \widetilde{S(G)}_z} F(x), \end{aligned} \qquad \text{for} \quad z \in E. \tag{21}$$

In the case of a binary picture F with X as the set of points having grey-level 1, the set of points of $\delta_G(F)$ and $\varepsilon_G(F)$ having grey-level 1 will be $X \oplus S(G)$ and $X \ominus S(G)$ respectively. In the case of grey-level pictures, the dilation and erosion of (21) are the well-known Max and Min filters given in [Goetcherian] and [Nakagawa-Rosenfeld] as generalization of "expansion" and "shrinking" on two-tone pictures.

If we consider grey-level pictures $E \to D$, where D is a closed segment in $\mathbf{R}$, then the examples given above are valid, provided that: (i) we assume that $\sup_{x \in S(G)} G(x) = 0$, (ii) in (17) and (19) we truncate all resulting grey-levels outside the bounds of D: in other words all grey-levels smaller than the lower bound of that segment are replaced by it, and all grey-levels larger than its upper bound are replaced by it. However, for dilations and erosions with a flat structuring picture (see (21)), no such identifications are necessary.

The latter example indicates that particular cases of dilations and erosions can be rather tricky. It is thus interesting to think about them without using complicated formulas. This shows the interest of the general framework of complete lattices, where important concepts can be presented with just a few symbols, without having to care about the interpretation of arithmetical expressions having out of bounds terms.

After all these examples, we will now state the main properties of dilations and erosions. By duality, it is sufficient to prove theorems about dilations only, since dual theorems about erosions follow immediately.

Proposition 2 [Serra-Th]. (*i*) *Dilations are increasing.*

(*ii*) $\mathcal{D}$ *is closed under composition and contains* 1.

(*iii*) $\mathcal{D}$ *is a dual Moore family.*

Proof. (*i*) follows from the fact that the relation $\preceq$ is determined by $\bigvee$. More precisely, for any $X, Y \in \mathcal{L}$, $X \preceq Y$ means that $Y = X \vee Y$, and so for a dilation δ we have $\delta(Y) = \delta(X) \vee \delta(Y)$, which means that $\delta(X) \preceq \delta(Y)$.

(*ii*) Consider two dilations δ, δ'. For any $\mathcal{T} \subseteq \mathcal{L}$,

$$[\delta\delta'](\bigvee \mathcal{T}) = \delta(\delta'(\bigvee \mathcal{T})) = \delta(\bigvee_{X \in \mathcal{T}} \delta'(X)) = \bigvee_{X \in \mathcal{T}} \delta(\delta'(X)),$$

and so $\delta\delta'$ is a dilation. Now clearly

$$1(\bigvee \mathcal{T}) = \bigvee \mathcal{T} = \bigvee_{X \in \mathcal{T}} X = \bigvee_{X \in \mathcal{T}} 1(X),$$

and so 1 is a dilation.

(*iii*) We must show that for any $\mathcal{Q} \subseteq \mathcal{D}$ (including a void one) $\bigvee \mathcal{Q}$ is a dilation, in other words that for any $\mathcal{T} \subseteq \mathcal{L}$,

$$[\bigvee \mathcal{Q}](\bigvee \mathcal{T}) = \bigvee_{X \in \mathcal{T}} ([\bigvee \mathcal{Q}](X)).$$

Indeed, if $\mathcal{Q}$ is empty, then $\bigvee \mathcal{Q} = \langle \mathbf{O} \rangle$, and it is clear that

$$\langle \mathbf{O} \rangle(\bigvee \mathcal{T}) = \mathbf{O} = \bigvee_{X \in \mathcal{T}} \mathbf{O} = \bigvee_{X \in \mathcal{T}} \langle \mathbf{O} \rangle(X).$$

On the other hand, if $\mathcal{Q}$ is not empty, then we have

$$\begin{aligned}
[\bigvee \mathcal{Q}](\bigvee \mathcal{T}) &= \bigvee_{\delta \in \mathcal{Q}} \delta(\bigvee \mathcal{T}) && \text{(by definition, see(6));} \\
&= \bigvee_{\delta \in \mathcal{Q}} (\bigvee_{X \in \mathcal{T}} \delta(X)) && \text{(since each } \delta \in \mathcal{Q} \text{ is a dilation);} \\
&= \bigvee_{X \in \mathcal{T}} (\bigvee_{\delta \in \mathcal{Q}} \delta(X)) && \text{(by the commutativity of } \textstyle\bigvee\text{);} \\
&= \bigvee_{X \in \mathcal{T}} ([\bigvee \mathcal{Q}](X)) && \text{(by (6)).} \blacksquare
\end{aligned}$$

The dual result concerning erosions is the following:

Proposition 2' [Serra-Th]. *(i) Erosions are increasing.*

(ii) $\mathcal{E}$ is closed under composition and contains 1.

(iii) $\mathcal{E}$ is a Moore family.

A particular consequence of Proposition 2/2' is that $\mathcal{D}$ and $\mathcal{E}$ are complete lattices. The lowest dilation is $\langle \mathbf{O} \rangle$, while the highest one fixes $\mathbf{O}$ and transforms every other picture into $\mathbf{I}$. Similarly, the highest erosion is $\langle \mathbf{I} \rangle$, while the lowest one fixes $\mathbf{I}$ and transforms every other picture into $\mathbf{O}$. Let us describe some simple dilations and erosions which can be defined on any lattice.

We know that every dilation fixes $\mathbf{O}$. Suppose now that we have a dilation δ such that for all $X \in \mathcal{L}$, $\delta(X)$ must be one of two fixed pictures; one of them is of course $\mathbf{O}$, and let us write Y for the other ($Y \neq \mathbf{O}$). Set

$$X = \bigvee \{Z \in \mathcal{L} \mid \delta(Z) = \mathbf{O}\}.$$

As δ is a dilation, we have $\delta(X) = \bigvee_{\delta(Z)=\mathbf{O}} \delta(Z) = \mathbf{O}$, and for every $Z \in \mathcal{L}$, $\delta(Z) = \mathbf{O}$ iff $Z \preceq X$. Since $\delta(Z) = Y \neq \mathbf{O}$ for some Z, we must have $X \neq \mathbf{I}$. Thus δ is the operator $\delta_{X,Y}$ defined by

$$\delta_{X,Y}(Z) = \begin{cases} \mathbf{O} & \text{if } Z \preceq X, \\ Y & \text{if } Z \not\preceq X, \end{cases} \tag{22}$$

with $X \neq \mathbf{I}$ and $Y \neq \mathbf{O}$. Conversely, for any $X, Y \in \mathcal{L}$ with $X \neq \mathbf{I}$ and $Y \neq \mathbf{O}$, the operator $\delta_{X,Y}$ defined in (22) is a dilation. Such a dilation is called a *dyadic* dilation, since its result can take only two distinct values. We define similarly *dyadic* erosions, that is those whose result can take only two distinct values. They are the operators $\varepsilon_{Y,X}$ defined for any $Y, X \in \mathcal{L}$ with $Y \neq \mathbf{O}$ and $X \neq \mathbf{I}$ by

$$\varepsilon_{Y,X}(Z) = \begin{cases} \mathbf{I} & \text{if } Z \succeq Y, \\ X & \text{if } Z \not\succeq Y, \end{cases} \tag{23}$$

Note that when $X = \mathbf{I}$ or $Y = \mathbf{O}$, the operators $\delta_{X,Y}$ and $\varepsilon_{Y,X}$ defined in (22) and (23) are still a dilation and an erosion, but they reduce then to null and universal operators $\langle \mathbf{O} \rangle$ and $\langle \mathbf{I} \rangle$. Dyadic dilations and erosions will be used as building blocks for the construction of various operators, for example in the proof of several of our results.

The next proposition is a generalization of a result of [Matheron-RSIG], and is due to [Serra-Th], where it was stated in the dual form (with erosions instead of dilations). We have also simplified its proof:

Proposition 3 [Serra-Th]. *Let $\beta \in \mathcal{O}$. Then the following two statements are equivalent:*

(i) β is increasing and $\beta(\mathbf{O}) = \mathbf{O}$.

(ii) β is the infimum of a set of dilations.

Proof. <u>(ii) implies (i)</u>: Let $\mathcal{Q} \subseteq \mathcal{L}$. First, we have

$$[\bigwedge \mathcal{Q}](\mathbf{O}) = \bigwedge_{\delta \in \mathcal{Q}} \delta(\mathbf{O}) = \bigwedge_{\delta \in \mathcal{Q}} \mathbf{O} = \mathbf{O}.$$

Second, Proposition 2 (i) implies that every $\delta \in \mathcal{Q}$ is increasing, and so by Proposition 1 (ii) $\bigwedge \mathcal{Q}$ is increasing.

<u>(i) implies (ii)</u>: Suppose that β is increasing and preserves $\mathbf{O}$. For any $B \in \mathcal{L}$, consider the two dilations $\delta_{\mathbf{O},\beta(B)}$ and $\delta_{B,\mathbf{I}}$ defined as follows (by (22)):

$$\delta_{\mathbf{O},\beta(B)}(Z) = \begin{cases} \mathbf{O} & \text{if } Z = \mathbf{O}, \\ \beta(B) & \text{if } Z \neq \mathbf{O}, \end{cases} \qquad \delta_{B,\mathbf{I}}(Z) = \begin{cases} \mathbf{O} & \text{if } Z \preceq B, \\ \mathbf{I} & \text{if } Z \not\preceq B. \end{cases}$$

Then their supremum $\delta_B = \delta_{\mathbf{O},\beta(B)} \vee \delta_{B,\mathbf{I}}$ is a dilation by Proposition 2 (iii). In fact, the two previous equalities imply that for $Z \in \mathcal{L}$ we have

$$\delta_B(Z) = \begin{cases} \mathbf{O} & \text{if } Z = \mathbf{O}, \\ \beta(B) & \text{if } \mathbf{O} \prec Z \preceq B, \\ \mathbf{I} & \text{if } Z \not\preceq B. \end{cases} \tag{24}$$

Let $\gamma = \bigwedge_{B \in \mathcal{L}} \delta_B$. We must show that $\gamma = \beta$.

First it is clear by (24) that $\delta_B(\mathbf{O}) = \mathbf{O}$ for any $B \in \mathcal{L}$, and so that $\gamma(\mathbf{O}) = \mathbf{O} = \beta(\mathbf{O})$. Take now $B, Z \in \mathcal{L}$ such that $Z \neq \mathbf{O}$. If $Z \preceq B$, then by (24) $\delta_B(Z) = \beta(B)$; as β is increasing, this implies that $\delta_B(Z) = \beta(B) \succeq \beta(Z)$. If $Z \not\preceq B$, then by (24) $\delta_B(Z) = \mathbf{I} \succeq \beta(Z)$. Thus $\delta_B(Z) \succeq \beta(Z)$ for any $B \in \mathcal{L}$, and as $\delta_Z(Z) = \beta(Z)$, we have $\gamma(Z) = \beta(Z)$. Hence $\gamma = \beta$ and so β is an infimum of dilations. ∎

We leave the statement of the dual result to the reader. Let us point to an interesting consequence of Proposition 3: Consider the particular case where $\mathcal{L}$ is the set of binary pictures on E. As explained after (21), a dilation assigns to a point p a grey-level equal to the supremum of grey-levels of points in a given neighborhood of p. Thus a non-constant increasing transform for binary pictures, being an infimum of dilations, assigns to a point p a grey-level equal to the infimum of suprema of grey-levels of points in certain neighborhoods of p. This implies in particular that every non-constant increasing function $\{0,1\}^E \to \{0,1\}$ is an infimum of suprema, or if E is finite, a minimum of maxima: we obtain in this way a well-known result in the theory of Boolean functions!

IV. Morphological duality

We said above that dilation and erosion are dual concepts from the lattice point of view. When we have an isomorphism between the complete lattice $\mathcal{L}$ and its dual (for example the complementation if $\mathcal{L}$ is the set of binary images on a set E), this isomorphism induces an isomorphism between the dual Moore family $\mathcal{D}, \bigvee$ of dilations and the Moore family $\mathcal{E}, \bigwedge$ of erosions. We will show below that for any complete lattice, we always have a duality between $\mathcal{D}$ and $\mathcal{E}$, which is not dependent upon a duality between complete lattices. Such a duality will be called the *morphological duality*.

Definition 2. Let $\eta, \theta \in \mathcal{O}$. Then we will write $\eta \perp \theta$ if for every $X, Y \in \mathcal{L}$, we have

$$\eta(X) \preceq Y \quad \Longleftrightarrow \quad X \preceq \theta(Y). \tag{25}$$

Thus $\perp$ is a relation on $\mathcal{O}$.

Note that (25) can be expressed in a dual form with $\succeq$ instead of $\preceq$:

$$\theta(Y) \succeq X \quad \Longleftrightarrow \quad Y \succeq \eta(X).$$

Thus $\eta \perp \theta$ in $\mathcal{L}, \preceq$ means that in the dual lattice $\mathcal{L}, \succeq$ we have $\theta \perp \eta$. Hence to every property of η and θ will correspond a dual property with θ and η inverted.

This relation $\perp$ will in fact be our morphological duality between dilations and erosions. We show first that it does not apply to other operators:

Lemma 4. *$\perp$ is a relation from $\mathcal{D}$ to $\mathcal{E}$; in other words, for any $\eta, \theta \in \mathcal{O}$ such that $\eta \perp \theta$, η is a dilation and θ is an erosion.*

Proof. We have only to show that η is a dilation. The fact that θ is an erosion follows then by duality.

As we have $\mathbf{O} \preceq \theta(\mathbf{O})$ anyway, the definition of $\perp$ implies that $\eta(\mathbf{O}) \preceq \mathbf{O}$, and so $\eta(\mathbf{O}) = \mathbf{O}$. Take now a non-empty $\mathcal{T} \subseteq \mathcal{L}$, and let Y be any element of $\mathcal{L}$. We obtain the following succession of equivalent statements:

$$\begin{aligned}
\bigvee_{X \in \mathcal{T}} \eta(X) &\preceq Y; \\
\forall X \in \mathcal{T}, \quad \eta(X) &\preceq Y \qquad \text{(by definition of } \textstyle\bigvee); \\
\forall X \in \mathcal{T}, \quad X &\preceq \theta(Y) \qquad \text{(by definition of } \perp); \\
\bigvee \mathcal{T} &\preceq \theta(Y) \qquad \text{(by definition of } \textstyle\bigvee); \\
\eta(\bigvee \mathcal{T}) &\preceq Y \qquad \text{(by definition of } \perp).
\end{aligned}$$

Thus $\bigvee_{X \in \mathcal{T}} \eta(X) \preceq Y$ iff $\eta(\bigvee \mathcal{T}) \preceq Y$. Taking successively $Y = \bigvee_{X \in \mathcal{T}} \eta(X)$ and $Y = \eta(\bigvee \mathcal{T})$, we obtain

$$\bigvee_{X \in \mathcal{T}} \eta(X) = \eta(\bigvee \mathcal{T}),$$

in other words η is a dilation. ∎

We can now establish that the relation $\perp$ between dilations and erosions is in fact a duality:

Theorem 5 [Serra-Th]. *The relation $\perp$ is a duality between $\mathcal{D}$ and $\mathcal{E}$, that is a bijection which reverses the ordering relation $\preceq$. In other words:*

(*i*) *For any dilation δ, there is exactly one erosion ε such that $\delta \perp \varepsilon$.*

(ii) *For any erosion ε, there is exactly one dilation δ such that $\delta \perp \varepsilon$.*

(iii) *Given two dilations δ, δ' and two erosions $\varepsilon, \varepsilon'$ such that $\delta \perp \varepsilon$ and $\delta' \perp \varepsilon'$, we have $\delta \preceq \delta'$ iff $\varepsilon \succeq \varepsilon'$.*

Moreover, given a dilation δ and an erosion ε such that $\delta \perp \varepsilon$, the following hold:

(iv) *For any $Y \in \mathcal{L}$,*

$$\varepsilon(Y) = \bigvee\{Z \in \mathcal{L} \mid \delta(Z) \preceq Y\}. \tag{26}$$

(v) *For any $X \in \mathcal{L}$,*

$$\delta(X) = \bigwedge\{Z \in \mathcal{L} \mid X \preceq \varepsilon(Z)\}. \tag{27}$$

Proof. Let us first prove *(iii)*. The following statements are equivalent:

$$\begin{array}{llll}
\delta \preceq \delta'; & & & \\
\forall X \in \mathcal{L}, \quad \delta(X) \preceq \delta'(X); & & & \\
\forall X, Y \in \mathcal{L}, \quad \delta'(X) \preceq Y & \Longrightarrow & \delta(X) \preceq Y; & \\
\forall X, Y \in \mathcal{L}, \quad X \preceq \varepsilon'(Y) & \Longrightarrow & X \preceq \varepsilon(Y) & \text{(by definition of } \perp); \\
\forall Y \in \mathcal{L}, \quad \varepsilon'(Y) \preceq \varepsilon(Y); & & & \\
\varepsilon' \preceq \varepsilon. & & &
\end{array}$$

Thus *(iii)* holds. We have now only to show *(i)* and *(iv)*, because *(ii)* and *(v)* follow then by duality.

We consider the dilation δ. Let us first show that there is at most one ε such that $\delta \perp \varepsilon$. Indeed, if $\delta \perp \varepsilon$ and $\delta \perp \varepsilon'$, then the definition of $\perp$ implies that for any $X, Y \in \mathcal{L}$,

$$X \preceq \varepsilon(Y) \iff \delta(X) \preceq Y \iff X \preceq \varepsilon'(Y).$$

Taking successively $X = \varepsilon(Y)$ and $X = \varepsilon'(Y)$, we obtain that $\varepsilon(Y) = \varepsilon'(Y)$. Hence $\varepsilon = \varepsilon'$ and so to δ corresponds at most one ε with $\delta \perp \varepsilon$.

Let us now show that the operator ε defined by *(iv)* satisfies $\delta \perp \varepsilon$. Take any $X, Y \in \mathcal{L}$, and let

$$\mathcal{P}_Y = \{Z \in \mathcal{L} \mid \delta(Z) \preceq Y\}.$$

We have $\varepsilon(Y) = \bigvee \mathcal{P}_Y$. If $\delta(X) \preceq Y$, then $X \in \mathcal{P}_Y$, and so $X \preceq \bigvee \mathcal{P}_Y = \varepsilon(Y)$. If $X \preceq \varepsilon(Y) = \bigvee \mathcal{P}_Y$, then the fact that δ is a dilation implies that

$$\delta(X) \preceq \delta(\bigvee \mathcal{P}_Y) = \bigvee_{Z \in \mathcal{P}_Y} \delta(Z);$$

now $\delta(Z) \preceq Y$ for every $Z \in \mathcal{P}_Y$ (by definition), and so $\delta(X) \preceq Y$. We have thus shown that for any $X, Y \in \mathcal{L}$, $\delta(X) \preceq Y$ iff $X \preceq \varepsilon(Y)$, in other words that $\delta \perp \varepsilon$.

Hence to the dilation δ corresponds a unique ε such that $\delta \perp \varepsilon$, and ε satisfies *(iv)*. Now ε is an erosion by Lemma 4, and so *(i)* and *(iv)* hold. ∎

Definition 3. The relation $\perp$ is called the *morphological duality*. Given a dilation δ and an erosion ε such that $\delta \perp \varepsilon$, we will say that ε is the *morphological dual* of δ, and vice versa. The morphological dual of a dilation δ will be written $\dot{\delta}$.

A consequence of Theorem 5 (*iii*) is that the dual of a supremum of dilations is the infimum of their respective duals, that the dual of $\langle \mathbf{O} \rangle$ is $\langle \mathbf{I} \rangle$, etc..

Let us now describe the duals of the examples of dilations given in the preceding section. It is easy to check that for any $X, Y \in \mathcal{L}$, the dilation $\delta_{X,Y}$ defined in (22) is the dual of the erosion $\varepsilon_{Y,X}$ defined in (23).

Consider next the case of binary pictures on E. We have a dilation δ given by (7), and we suppose that for each $x \in E$,

$$\delta(\{x\}) = W(x), \tag{28}$$

where $W(x)$ is the window associated to x. We define the *dual* windows $W^*(x)$ as follows:

$$y \in W^*(x) \quad \iff \quad x \in W(y) \qquad \text{for every} \quad x, y \in E. \tag{29}$$

Now, for $x, y \in E$, we have $x \in \mathbf{C}\big(W^*(y)\big)$ iff $x \notin W^*(y)$, iff $y \notin W(x)$, iff $W(x) \subseteq \mathbf{C}(\{y\})$, iff $\delta(\{x\}) \subseteq \mathbf{C}(\{y\})$, iff $\{x\} \subseteq \dot{\delta}\big(\mathbf{C}(\{y\})\big)$, iff $x \in \dot{\delta}\big(\mathbf{C}(\{y\})\big)$. Thus $\mathbf{C}\big(W^*(y)\big) = \dot{\delta}\big(\mathbf{C}(\{y\})\big)$, and so by (10) $\dot{\delta}$ satisfies the following for evey $X \subseteq E$:

$$\dot{\delta}(X) = \bigcap_{y \in \mathbf{C}(X)} \dot{\delta}\big(\mathbf{C}(\{y\})\big) = \bigcap_{y \in \mathbf{C}(X)} \mathbf{C}\big(W^*(y)\big). \tag{30}$$

Thus $\dot{\delta}(X)$ is obtained by expanding each point y of $\mathbf{C}(X)$ to $W^*(y)$.

In the case where all windows are translates of a structuring element B, in other words when each $W(x) = B_x$, we have $\delta(X) = X \oplus B$ (see (9)). Then it is easy to see that for every $y \in E$ we have $W^*(y) = \widetilde{B}_y$, where $\widetilde{B}$ is defined by (15). Hence (11) and (30) imply that $\dot{\delta}(X) = X \ominus \widetilde{B}$. Note that we do not need to use (30): we can directly check by (9) and (11) that the operations $\oplus B$ and $\ominus \widetilde{B}$ satisfy (25), in other words that for any $X, Y, B \subseteq E$ we have:

$$X \oplus B \subseteq Y \quad \iff \quad X \subseteq Y \ominus \widetilde{B}. \tag{31}$$

Consider finally grey-level pictures $E \to \overline{\mathbf{R}}$. Given a structuring function G (in other words a map $S(G) \to \mathbf{R}$), the *dual structuring function* is the map $\widetilde{G} : \widetilde{S(G)} \to \mathbf{R} : z \mapsto G(-z)$. Recall the definition of the dilation and erosion of pictures by a structuring function in (16), (17), (18), and (19). Then the morphological dual of δ_G is $\varepsilon_{\widetilde{G}}$; by (19) we have

$$\varepsilon_{\widetilde{G}}(F)(z) = \inf_{x \in S(\widetilde{G})_z} \big(F(x) - \widetilde{G}(z - x)\big) = \inf_{x \in S(G)_z} \big(F(x) - G(x - z)\big) \qquad \text{for} \quad z \in E. \tag{32}$$

To show that $\varepsilon_{\widetilde{G}} = \dot{\delta}_G$, we have only to remark that for any two pictures $A, B \in \mathcal{L}$ the following statements are equivalent:

$\delta_G(A) \preceq B$.

$\forall y \in E, \quad \sup_{x \in \widetilde{S(G)}_y} \big(A(x) + G(y-x)\big) = \delta_G(A)(y) \leq B(y)$.

$\forall y \in E, \forall x \in \widetilde{S(G)}_y, \quad A(x) + G(y-x) \leq B(y)$.

$\forall x \in E, \forall y \in S(G)_x, \quad A(x) \leq B(y) - G(y-x)$
(since for $x, y \in E$, $x \in \widetilde{S(G)}_y$ iff $y \in S(G)_x$.)

$\forall x \in E, \quad A(x) \leq \inf_{y \in S(G)_x} \big(B(y) - G(y-x)\big) = \varepsilon_{\widetilde{G}}(B)(x)$.

$A \preceq \varepsilon_{\widetilde{G}}(B)$.

In particular, when the structuring function G is flat (see (20)), the dilation by G is defined by $\delta_G(F)(z) = \sup_{x \in \widetilde{S(G)}_z} F(x)$ and its morphological dual is defined by $\dot{\delta}_G(F)(z) = \varepsilon_{\widetilde{G}}(F)(z) = \inf_{x \in S(G)_z} F(x)$ (see (21)).

V. Openings and closings

Beside dilations and erosions, two other types of operations are important: openings and closings. They were first introduced in the framework of topology, and later extended to subsets of arbitrary sets. The concept of a closing on the set of parts of a set is due to Moore (see [Birkhoff]), and the opening represents the dual concept.

For any operator β, write β^2 for $\beta\beta$.

Definition 4. Let $\beta \in \mathcal{O}$. Then we say that:

(a) β is *idempotent* if $\beta^2 = \beta$.

(b) β is *extensive* if for every $X \in \mathcal{L}$, $X \preceq \beta(X)$.

(c) β is *anti-extensive* if for every $X \in \mathcal{L}$, $X \succeq \beta(X)$.

(d) β is a *closing* if it is extensive, increasing, and idempotent.

(e) β is an *opening* if it is anti-extensive, increasing, and idempotent.

We will write $\mathcal{F}$ for the set of closings and $\mathcal{A}$ for the set of openings (cfr. the Latin roots *ferm-* and *aper-*). Closings will be written $\varphi, \varphi', \varphi_1$, etc., while openings will be written $\alpha, \alpha', \alpha_1$, etc..

It is clear that extensivity and anti-extensivity, closings and openings, constitute pairs of dual concepts from the complete lattice point of view (see Section II). Hence to every statement on openings corresponds a dual statement on closings, and vice versa. Similarly to every example of opening corresponds a dual example of closing.

Let us indeed give such examples. If $\mathcal{L}$ is the set of subsets of an Euclidean space, then the topological operations of taking the interior of a set is an opening. On the other hand the topological closure and the convex hull are closings. If $\mathcal{L}$ is the set of subsets of a vector space, the operation of taking the vector subspace generated by a set is a closing.

We can also build openings and closings from other increasing operators, such as dilations and erosions:

Lemma 6. *Let δ be a dilation, and let $\beta = \delta \vee 1$. Then β is a closing iff $\delta^2 \preceq \beta$.*

Proof. Indeed, it is clear that β is extensive; it is increasing since 1 and δ are increasing. Now we have:

$$\begin{aligned}\beta^2 &= [\delta \vee 1]\beta;\\ &= [\delta\beta] \vee [1\beta] = [\delta[\delta \vee 1]] \vee [\delta \vee 1] \qquad \text{(by definition of } \vee \text{ on } \mathcal{O});\\ &= [[\delta\delta] \vee [\delta 1]] \vee [\delta \vee 1] = \delta^2 \vee \delta \vee 1 = \delta^2 \vee \beta \qquad \text{(since } \delta \text{ is a dilation)};\end{aligned}$$

Therefore β is a closing iff $\beta^2 = \beta$, iff $\delta^2 \vee \beta = \beta$, iff $\delta^2 \preceq \beta$. ∎

For example, if δ is a symmetry in the Euclidean space E, then φ associates to every subset X of E the smallest symmetric subset of E containing X, namely $X \cup \delta(X)$. Again, there is a dual version of Lemma 6 producing an opening from an erosion.

In [Serra-Th] an *annular opening* is defined on the set of subsets of a space E. Take a *symmetric* dilation δ, in other words a dilation δ such that for every $x, y \in E$, $y \in \delta(\{x\})$ iff $x \in \delta(\{y\})$; then $\alpha = \delta \wedge 1$ is an opening (we will show it later). Generally one takes δ to be the dilation $\oplus B$ (see (9)) by a *symmetric* structuring element B (in other words such that $B = \widetilde{B}$, see (15)), with the further condition that B does not contain the origin (B is like a ring). For example, if B is the set of points whose distance to the origin is comprised between a and b (where $0 < a < b$), then α will remove from a set X small isolated clusters, namely all points of X whose distance to other points of X is either smaller than a or larger than b.

We will generalize the annular opening $\delta \wedge 1$ to our general framework. We have first a simple criterion for obtaining an opening of the form $\eta \wedge 1$:

Lemma 7. *Let η be an increasing operator, and let $\beta = \eta \wedge 1$. Then β is an opening iff $\beta \preceq \eta\beta$.*

Proof. It is easy to see that β is anti-extensive and increasing (see the proof of Lemma 6). Now

$$\beta^2 = [\eta \wedge 1]\beta = [\eta\beta] \wedge \beta.$$

Thus β is an opening iff $\beta^2 = \beta$, iff $[\eta\beta] \wedge \beta = \beta$, iff $\beta \preceq \eta\beta$. ∎

The next result will allow a generalization of Serra's annular opening:

Lemma 8. *Let η be an increasing operator, let $A, B \in \mathcal{L}$, and let $\beta = \eta \wedge 1$. Consider the following condition:*

$(*)$ $A \preceq \beta(B)$ and there is some $C \preceq B$ such that $A \preceq \eta(C)$ and $C \preceq \eta(A)$.

If A and B satisfy $(*)$, then $A \preceq \eta\beta(B)$.

Proof. Take $B \in \mathcal{L}$ and $A \preceq \beta(B) = \eta(B) \wedge B$ satisfying the condition $(*)$. Then there is some $C \preceq B$ such that $A \preceq \eta(C)$ and $C \preceq \eta(A)$. Now $A \preceq B$ (since $\beta(B) \preceq B$), and η is increasing: hence $\eta(A) \preceq \eta(B)$, and as $C \preceq \eta(A)$, we have $C \preceq \eta(B)$; but we have also $C \preceq B$, and so $C \preceq \eta(B) \wedge B = \beta(B)$. As η is increasing, this implies that $\eta(C) \preceq \eta\beta(B)$, and as $A \preceq \eta(C)$, we get $A \preceq \eta\beta(B)$. ∎

Corollary 9. *Let η be an increasing operator and $\beta = \eta \wedge 1$. Suppose that for every $B \in \mathcal{L}$ there exists some subset $\mathcal{S}_B$ of $\mathcal{L}$ such that:*

(i) $\beta(B) = \bigvee \mathcal{S}_B$;

(ii) for every $A \in \mathcal{S}_B$, A and B satisfy the condition $(*)$ of Lemma 8.

Then β is an opening.

Proof. Take any $B \in \mathcal{L}$. For any $A \in \mathcal{S}_B$, (ii) and Lemma 8 imply that $A \preceq \eta\beta(B)$. Then by (i) we have $\beta(B) = \bigvee \mathcal{S}_B \preceq \eta\beta(B)$. As this holds for any B, it means that $\beta \preceq \eta\beta$. Thus β is an opening by Lemma 7. ∎

We must now explain why Corollary 9 generalizes the annular opening. Let δ be a symmetric dilation on the set of parts of the space E, and let $\alpha = \delta \wedge 1$. We will show that δ and α satisfy the hypothesis of Corollary 9, and so α will be an opening.

For any $B \subseteq E$ we take $\mathcal{S}_B$ to be the set of singletons $\{p\}$ for $p \in \alpha(B) = \delta(B) \cap B$. It is clear that $\alpha(B) = \bigcup \mathcal{S}_B$. Let us show that B and any $A = \{p\} \in \mathcal{S}_B$ satisfy the condition $(*)$ of Lemma 8. Indeed, as δ is a dilation, $\delta(B)$ is the union of all $\delta(\{q\})$ for $q \in B$. As $p \in \delta(B)$, there is some $q \in B$ such that $p \in \delta(\{q\})$, and by the symmetry of δ, we have also $q \in \delta(\{p\})$. For $A = \{p\}$ and $C = \{q\}$, we have $A \subseteq \alpha(B)$, $C \subseteq B$, $A \subseteq \delta(C)$ and $C \subseteq \delta(A)$, in other words the condition $(*)$.

Note that it is possible to show (using the axiom of choice) that the condition $(*)$ is satisfied for every $B \subseteq E$ and $A \subseteq \alpha(B)$.

One can also define an annular opening in the case of grey-level pictures $E \to \overline{\mathbf{R}}$, where $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$. Consider a structuring function G (see Section III); assume that G has a symmetric support (in other words that $\widetilde{S(G)} = S(G)$), and that

$$G(u) + G(-u) \geq 0 \qquad \text{for any} \quad u \in S(G). \tag{33}$$

Then $\delta_G \wedge 1$ is an opening. To show this, we have only to check the hypothesis of Corollary 9.

Let B be any picture. For any $z \in E$ and $y \in \widetilde{S(G)}_z = S(G)_z$, we define the picture A_{yz} as follows:

$$A_{yz}(z) = \min\bigl(B(z), B(y) + G(z - y)\bigr),$$
$$A_{yz}(x) = -\infty \qquad \text{for} \quad x \neq z.$$

If we define

$$A_z = \bigvee_{y \in \widetilde{S(G)}_z} A_{yz},$$

then we have (see (17)):

$$
\begin{aligned}
A_z(z) &= \sup_{y \in \widetilde{S(G)}_z} A_{yz}(z) = \sup_{y \in \widetilde{S(G)}_z} \min\bigl(B(z), B(y) + G(z-y)\bigr) \\
&= \min\bigl(B(z), \sup_{y \in \widetilde{S(G)}_z} B(y) + G(z-y)\bigr) = \min\bigl(B(z), \delta_G(B)(z)\bigr) = [\delta_G \wedge 1](B)(z),
\end{aligned}
$$

and $A_z(x) = -\infty$ for $x \neq z$. It follows then that

$$
\bigvee_{z \in E, y \in \widetilde{S(G)}_z} A_{yz} = \bigvee_{z \in E} A_z = [\delta_G \wedge 1](B).
$$

Thus $\mathcal{S}_B = \{A_{yz} \mid z \in E, y \in \widetilde{S(G)}_z\}$ satisfies condition (i) of Corollary 9. Let us show that it satisfies also condition (ii), in other words that B and every A_{yz} satisfy condition $(*)$ of Lemma 8.

As $[\delta_G \wedge 1](B)$ is the supremum of all pictures A_{yz}, it is clear that $A_{yz} \preceq [\delta_G \wedge 1](B)$. We define the picture C_{yz} as follows:

$$
\begin{aligned}
C_{yz}(y) &= \min\bigl(B(z) - G(z-y), B(y)\bigr), \\
C_{yz}(x) &= -\infty \qquad \text{for} \quad x \neq y.
\end{aligned}
$$

Clearly $C_{yz}(y) \leq B(y)$; as $C_{yz}(x) = -\infty$ for $x \neq y$, this means that $C_{yz} \preceq B$. By definition of A_{yz} and C_{yz} we have $A_{yz}(z) = C_{yz}(y) + G(z-y)$. Thus

$$
A_{yz}(z) = C_{yz}(y) + G(z-y) \leq \sup_{u \in \widetilde{S(G)}_z} C_{yz}(u) + G(z-u) = \delta_G(C_{yz})(z),
$$

and as $A_{yz}(x) = -\infty$ for $x \neq z$, this means that $A_{yz} \preceq \delta_G(C_{yz})$. By (33) we have $G(z-y) + G(y-z) \geq 0$, and as $A(z) = C_{yz}(y) + G(z-y)$, we get

$$
C_{yz}(y) \leq C_{yz}(y) + G(z-y) + G(y-z) = A_{yz}(z) + G(y-z).
$$

As $S(G)$ is symmetric, we have $z \in \widetilde{S(G)}_y$, and so the latter equation implies that

$$
C_{yz}(y) \leq A_{yz}(z) + G(y-z) \leq \sup_{u \in \widetilde{S(G)}_y} A_{yz}(u) + G(y-u) = \delta_G(A_{yz})(z),
$$

and as $C_{yz}(x) = -\infty$ for $x \neq y$, this means that $C_{yz} \preceq \delta_G(A_{yz})$. Hence $A_{yz} \preceq [\delta_G \wedge 1](B)$, $C_{yz} \preceq B$, $A_{yz} \preceq \delta_G(C_{yz})$, and $C_{yz} \preceq \delta_G(A_{yz})$, in other words condition $(*)$ is satisfied.

Therefore $\delta_G \wedge 1$ is an opening by Corollary 9. If we take the particular case of (33) where $G(u) = 0$ for every $u \in S(G)$, and restrict δ_G to the binary case, then δ_G reduces to a symmetric dilation on the set of parts of E, and we get the original definition of an annular opening considered above.

Let us illustrate the concept of annular opening in the case of pictures $\mathbf{R} \to \overline{\mathbf{R}}$. We define a structuring function G as follows: take $a, b, s \in \mathbf{R}$ such that $0 < a < b$ and $s > 0$, let $S(G)$ be the set of points $x \in \mathbf{R}$ such that $a \leq |x| \leq b$, and set $G(x) = sx$ for $x \in S(G)$. Given a picture $F : \mathbf{R} \to \overline{\mathbf{R}}$, the transformed picture $F' = [\delta_G \wedge 1](F)$ can be understood as follows: for any $x \in \mathbf{R}$ and $u \in \overline{\mathbf{R}}$, we have $F'(x) \geq u$ iff there exists $x' \in \mathbf{R}$ such that $a \leq |x' - x| \leq b$, $F(x) \geq u$, and $F(x') + s(x - x') \geq u$. A visual interpretation is given in Figure 2.

The examples of openings and closings given above are not the only ones. In the next section, we will show two ways in which openings or closings can be built by combining dilations and erosions, and we will illustrate them with examples related to those given in Section III for dilations and erosions.

Let us now give the main properties and characterizations of openings and closings. The first result is somewhat analogous to Proposition 2 in the case of dilations. We restrict ourselves to openings, the dual statement concerning closings is left to the reader.

Proposition 10 [Serra-Th]. *$\mathcal{A}$ is a dual Moore family, and it has 1 as supremum.*

Proof. It is obvious that $\langle \mathbf{O} \rangle$ is an opening (that it is anti-extensive, increasing, and idempotent).

Consider now a non-empty subset $\mathcal{Q}$ of $\mathcal{A}$. We must show that $\beta = \bigvee \mathcal{Q}$ is an opening. Let $X \in \mathcal{L}$. For every $\alpha \in \mathcal{Q}$, we have $\alpha(X) \preceq X$, since α is anti-extensive. Therefore

$$\beta(X) = \bigvee_{\alpha \in \mathcal{Q}} \alpha(X) \preceq X,$$

and so β is anti-extensive.

By Proposition 1 (ii), β is increasing.

For every $\alpha \in \mathcal{Q}$, as $\alpha \preceq \beta$, we have $\alpha\beta \preceq \beta\beta$, and as α is increasing, we have also $\alpha\alpha \preceq \alpha\beta$. Thus $\alpha\alpha \preceq \beta\beta$, and as α is idempotent, $\alpha \preceq \beta\beta$. Therefore

$$\beta = \bigvee_{\alpha \in \mathcal{Q}} \alpha \preceq \beta\beta.$$

On the other hand, as β is anti-extensive, we have $\beta\beta \preceq \beta$. Combining both inequalities, we obtain $\beta\beta = \beta$, in other words β is idempotent.

We have thus shown that $\langle \mathbf{O} \rangle \in \mathcal{A}$, and that for a nonvoid $\mathcal{Q} \subseteq \mathcal{A}$, $\beta = \bigvee \mathcal{Q}$ is an opening. Hence $\mathcal{A}$ is a dual Moore family.

It is obvious that 1 is anti-extensive, increasing, and idempotent. It is thus an opening, and it is clear that it is above any other opening. It is thus the supremum of $\mathcal{A}$. ∎

Unlike in the case of dilations, we cannot prove that the composition of two openings is an opening: this composition is well anti-extensive and increasing, but not necessarily idempotent. However, if we have two openings α and α' such that $\alpha \preceq \alpha'$, then $\alpha\alpha' = \alpha'\alpha = \alpha$, thanks to the following result:

Proposition 11. *Let α be an opening, and let $\beta \in \mathcal{O}$ such that $\alpha \preceq \beta \preceq 1$. Then $\alpha\beta = \beta\alpha = \alpha$.*

Proof. As α is increasing and $\alpha \preceq \beta \preceq 1$, we have

$$\alpha\alpha \preceq \beta\alpha \preceq 1\alpha \qquad \text{and} \qquad \alpha\alpha \preceq \alpha\beta \preceq \alpha 1.$$

As α is idempotent, this means that $\alpha \preceq \beta\alpha \preceq \alpha$ and $\alpha \preceq \alpha\beta \preceq \alpha$, in other words $\beta\alpha = \alpha\beta = \alpha$. ∎

Conversely, given two openings α and α' such that $\alpha\alpha' = \alpha$, we must have $\alpha \preceq \alpha'$; indeed, as α is anti-extensive, we have $\alpha \preceq 1$, and so $\alpha = \alpha\alpha' \preceq 1\alpha' = \alpha'$.

Moore introduced closings on the complete lattice formed by the set of parts of a set E, and showed that there is a bijection between closings and Moore families in E (see [Birkhoff]). Dually, there is a bijection between openings and dual Moore families in E. We can generalize Moore's idea to any complete lattice. We will consider the case of openings, and the following result has of course a dual concerning closings:

Theorem 12. *Let $\mathcal{B} \subseteq \mathcal{L}$ and $\alpha \in \mathcal{O}$. Then the following two statements are equivalent:*

(i) α is an opening and $\mathcal{B} = \{B \in \mathcal{L} \mid \alpha(B) = B\}$.

(ii) $\mathcal{B}$ is a dual Moore family and for every $X \in \mathcal{L}$, $\alpha(X) = \bigvee\{B \in \mathcal{B} \mid B \preceq X\}$.

Proof. For any $X \in \mathcal{L}$, let $\mathcal{B}_X = \{B \in \mathcal{B} \mid B \preceq X\}$.

<u>(*i*) implies (*ii*)</u>: As α is anti-extensive, $\alpha(\mathbf{O}) \preceq \mathbf{O}$, and so $\mathbf{O} \in \mathcal{B}$. Let $\mathcal{T}$ be a nonvoid subset of $\mathcal{B}$, and let $T = \bigvee \mathcal{T}$. For any $B \in \mathcal{T}$, we have $B \preceq T$, and as α is increasing, $B = \alpha(B) \preceq \alpha(T)$. Thus $T = \bigvee \mathcal{T} \preceq \alpha(T)$; as α is anti-extensive, we have $\alpha(T) \preceq T$. Combining both inequalities, we get $T = \alpha(T)$, and so $T \in \mathcal{B}$. Hence $\mathcal{B}$ is a dual Moore family.

Let $X \in \mathcal{L}$. As α is idempotent, $\alpha(X) \in \mathcal{B}$, and as α is anti-extensive, $\alpha(X) \preceq X$. Thus $\alpha(X) \in \mathcal{B}_X$, and so we have $\alpha(X) \preceq \bigvee \mathcal{B}_X$. On the other hand, for any $B \in \mathcal{B}_X$ we have $B \preceq X$, and as α is increasing, $B = \alpha(B) \preceq \alpha(X)$; hence $\bigvee \mathcal{B}_X \preceq \alpha(X)$. Combining both inequalities, we get $\bigvee \mathcal{B}_X = \alpha(X)$.

<u>(*ii*) implies (*i*)</u>: Let $X, Y \in \mathcal{L}$. If $X \preceq Y$, then $\mathcal{B}_X \preceq \mathcal{B}_Y$, and so $\alpha(X) = \bigvee \mathcal{B}_X \preceq \bigvee \mathcal{B}_Y = \alpha(Y)$. Thus α is increasing. For any $B \in \mathcal{B}_X$ we have $B \preceq X$, and so $\alpha(X) = \bigvee \mathcal{B}_X \preceq X$. Thus α is anti-extensive.

For any $B \in \mathcal{B}$, we have $B \in \mathcal{B}_B$, and so $\alpha(B) = \bigvee \mathcal{B}_B \succeq B$. As α is anti-extensive, this implies that $\alpha(B) = B$.

As $\mathcal{B}$ is a dual Moore family, we have $\alpha(X) = \bigvee \mathcal{B}_X \in \mathcal{B}$, and the previous paragraph implies then that $\alpha\big(\alpha(X)\big) = \alpha(X)$. Thus α is idempotent. Therefore α is an opening.

For any $B \in \mathcal{B}$, we have $B = \alpha(B)$. On the other hand, for any $X \in \mathcal{L}$ we have $\alpha(X) \in \mathcal{B}$, and so if $X = \alpha(X)$, then $X \in \mathcal{B}$. Hence $\mathcal{B}$ is the set of all $B \in \mathcal{L}$ such that $\alpha(B) = B$. ∎

For any $B \in \mathcal{L}$ and $\beta \in \mathcal{O}$, we will say that B is *an invariant of* β if $\beta(B) = B$. Thanks to (i), the opening α uniquely determines the set $\mathcal{B}$ of its invariants, and by (ii) the dual Moore family $\mathcal{B}$ uniquely determines the opening α. Hence we have the following:

Corollary 13. *There is a bijection between dual Moore families in $\mathcal{L}$ and openings in $\mathcal{O}$. An opening α and a dual Moore family $\mathcal{B}$ which correspond under this bijection define each other as follows:*

$$\mathcal{B} = \{B \in \mathcal{L} \mid \alpha(B) = B\}.$$
$$\forall X \in \mathcal{L}, \quad \alpha(X) = \bigvee \{B \in \mathcal{B} \mid B \preceq X\}.$$

For example the opening $\langle \mathbf{O} \rangle$ has $\mathbf{O}$ as unique invariant, and so it corresponds to the dual Moore family $\{\mathbf{O}\}$; every element of $\mathcal{L}$ is an invariant of the opening 1, and so it corresponds to the dual Moore family $\mathcal{L}$. Given two openings α and α' having to two dual Moore families $\mathcal{B}$ and $\mathcal{B}'$ as respective sets of invariants, we have $\alpha \preceq \alpha'$ iff $\mathcal{B} \subseteq \mathcal{B}'$. Given a set $\mathcal{Q}$ of openings to which corresponds a set $\mathcal{M}$ of dual Moore families, $\bigvee \mathcal{Q}$ corresponds to the smallest dual Moore family containing $\bigcup \mathcal{M}$.

For the annular opening illustrated in Figure 2, the invariants are all functions F such that for every $x \in \mathbf{R}$, there is some $y \in \mathbf{R}$ such that $a \leq |y - x| \leq b$ and $F(x) - F(y) \leq s(x - y)$. They form a Moore family generated by all functions $F_{x,y,u}$ (where $x, y \in \mathbf{R}$, $a \leq |y - x| \leq b$, and $u \in \overline{\mathbf{R}}$), which are defined by

$$F_{x,y,u}(x) = u;$$
$$F_{x,y,u}(y) = u + s(y - x);$$
$$F_{x,y,u}(z) = -\infty \qquad \text{for} \quad z \neq x, y.$$

VI. Building openings and closings from dilations and erosions

Morphological openings and closings were introduced by Matheron [Matheron-RSIG] for the complete lattice of subsets of a Euclidean space: given a structuring element B, the dilation $\delta = \oplus B$, and its morphological dual erosion $\dot{\delta} = \ominus \widetilde{B}$, $\delta\dot{\delta}$ is an opening and $\dot{\delta}\delta$ is a closing. Given a set X, the transform $(X \ominus \widetilde{B}) \oplus B$ of X by that opening is the union of all translates of B contained in X. This result can easily be extended to the general case. Again, we restrict ourselves to openings:

Theorem 14. *Let δ be a dilation and $\dot{\delta}$ its morphological dual. Then $\delta\dot{\delta}$ is an opening having $\{\delta(Z) \mid Z \in \mathcal{L}\}$ as set of invariants.*

Proof. We recall (26): for any $W \in \mathcal{L}$,

$$\dot{\delta}(W) = \bigvee \{Z \in \mathcal{L} \mid \delta(Z) \preceq W\}.$$

As δ is a dilation, this implies that

$$\delta\dot{\delta}(W) = \delta\big(\bigvee \{Z \in \mathcal{L} \mid \delta(Z) \preceq W\}\big) = \bigvee \{\delta(Z) \mid Z \in \mathcal{L}, \delta(Z) \preceq W\}.$$

If we set $\mathcal{B} = \{\delta(Z) \mid Z \in \mathcal{L}\}$, then the previous equality becomes

$$\delta\dot{\delta}(W) = \bigvee\{B \in \mathcal{B} \mid B \preceq W\}. \tag{34}$$

Let us now show that $\mathcal{B}$ is a dual Moore family. For every $\mathcal{C} \subseteq \mathcal{B}$, there is some $\mathcal{T} \subseteq \mathcal{L}$ such that $\mathcal{C} = \{\delta(Z) \mid Z \in \mathcal{T}\}$. As δ is a dilation, we have

$$\bigvee \mathcal{C} = \bigvee\{\delta(Z) \mid Z \in \mathcal{T}\} = \delta(\bigvee\{Z \mid Z \in \mathcal{T}\}),$$

in other words $\bigvee \mathcal{C} \in \mathcal{B}$. Thus $\mathcal{C}$ is a dual Moore family, and as we have (34), Theorem 12 implies that $\delta\dot{\delta}$ is an opening having $\mathcal{B}$ as set of invariants. ∎

The dual result states that $\dot{\delta}\delta$ is a closing having $\{\dot{\delta}(Z) \mid Z \in \mathcal{L}\}$ as set of invariants.

In the above proof we showed that

$$\delta\dot{\delta}(W) = \bigvee\{\delta(Z) \mid Z \in \mathcal{L}, \delta(Z) \preceq W\}. \tag{35}$$

We have also the dual equality:

$$\dot{\delta}\delta(W) = \bigwedge\{\dot{\delta}(Z) \mid Z \in \mathcal{L}, W \preceq \dot{\delta}(Z)\}. \tag{36}$$

We can also apply (26) and (27) directly with $X = \dot{\delta}(W)$ and $Y = \delta(W)$, and so we get:

$$\begin{aligned} \delta\dot{\delta}(W) &= \bigwedge\{Z \in \mathcal{L} \mid \dot{\delta}(W) \preceq \dot{\delta}(Z)\}; \\ \dot{\delta}\delta(W) &= \bigvee\{Z \in \mathcal{L} \mid \delta(Z) \preceq \delta(W)\}. \end{aligned} \tag{37}$$

Definition 5. Given a dilation δ and its morphological dual erosion $\dot{\delta}$, the opening $\delta\dot{\delta}$ is called a *morphological opening*, while the closing $\dot{\delta}\delta$ is called a *morphological closing*.

Note that it is possible to prove Theorem 14 directly, without using Theorem 12. As δ and $\dot{\delta}$ are increasing, so is $\delta\dot{\delta}$. For any $W \in \mathcal{L}$, applying (25) for $X = \dot{\delta}(W)$ and $Y = W$, we obtain that $\delta\dot{\delta}(W) \preceq W$ iff $\dot{\delta}(W) \preceq \dot{\delta}(W)$, in other words always. Thus $\delta\dot{\delta}$ is anti-extensive. To show that $\delta\dot{\delta}$ is idempotent, we rely on the following:

Lemma 15. *Given a dilation δ and its morphological dual $\dot{\delta}$, we have $\dot{\delta}\delta\dot{\delta} = \dot{\delta}$ and $\delta\dot{\delta}\delta = \delta$.*

Proof. We know that $\delta\dot{\delta}$ is anti-extensive. By duality, $\dot{\delta}\delta$ is extensive. Combining the three facts that $\delta\dot{\delta} \preceq 1$, $1 \preceq \dot{\delta}\delta$, and $\dot{\delta}$ is increasing, we obtain:

$$\dot{\delta} = 1\dot{\delta} \preceq [\dot{\delta}\delta]\dot{\delta} = \dot{\delta}[\delta\dot{\delta}] \preceq \dot{\delta}1 = \dot{\delta}.$$

Thus we have $\dot{\delta}\delta\dot{\delta} = \dot{\delta}$ and the fact that $\delta\dot{\delta}\delta = \delta$ follows by duality. ∎

It follows that then that $\delta\dot{\delta}\delta\dot{\delta} = \delta\dot{\delta}$, in other words that $\delta\dot{\delta}$ is idempotent. Therefore $\delta\dot{\delta}$ is an opening. For any $X \in \mathcal{L}$, the fact that $\delta\dot{\delta}\delta(X) = \delta(X)$ means that $\delta(X)$ is an invariant of $\delta\dot{\delta}$; conversely, given an invariant B of $\delta\dot{\delta}$, we have $B = \delta(X)$ for $X = \dot{\delta}(B)$. Hence the set of invariants of $\delta\dot{\delta}$ is the set of all $\delta(X)$ for $X \in \mathcal{L}$.

Let us illustrate morphological openings and closings with examples derived from those of dilations and erosions given in Section III. We consider first the complete lattice formed by the set of subsets of a space E. Let B be a structuring element. It is clear from (9) that the dilation of a set by B is a union of translates B_x of B. Therefore (35) implies that for every $W \subseteq E$, the result $(W \ominus \widetilde{B}) \oplus B$ of the opening of W by B is the union of all translates B_x of B included in W. This opening removes from W all portions of it which are too small (or too narrow) to contain a translate of B. The closing of W by B gives $(W \oplus B) \ominus \widetilde{B}$, which is equal to $\mathbf{C}\big((\mathbf{C}(W) \ominus B) \oplus \widetilde{B}\big)$; in other words it corresponds to an opening of the complement of W by $\widetilde{B}$, and so $(W \oplus B) \ominus \widetilde{B}$ is the complement of the union of all translates $\widetilde{B}_x$ of $\widetilde{B}$ which do not intersect W. This closing fills gaps in W which are too small (or too narrow) to contain a translate of $\widetilde{B}$.

Write α_B and φ_B for the opening and the closure by the structuring element B. In the case of a digital plane E, when B is a 3×3 neighborhood, these two operations have been used for the deletion of 1-pixel thick portions of a picture W or of its complement.

If E is a Euclidean space, let φ_{ch} be the operation of taking the convex hull; we mentioned in the preceding section that φ_{ch} is a closing. It is shown in [Serra-IAMM] that if B is a bounded set, then $\varphi_B \preceq \varphi_{ch}$. (In fact, one can even show that φ_{ch} is the supremum of such closings). By the dual version of Proposition 11, this implies that $\varphi_B\varphi_{ch} = \varphi_{ch}\varphi_B = \varphi_{ch}$. In particular, when W is convex, it is invariant under the closing by B.

Consider next grey-level pictures $E \to \overline{\mathbf{R}}$ (where $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$), with dilations and erosions by structuring functions. Given a structuring function $G : S(G) \to \mathbf{R}$ (where $S(G) \subseteq E$), (16) implies that for every $X \in \mathcal{L}$, $\delta_G(X)$ is a supremum of pictures of the form $Ext_{-\infty}\big(\sigma_u(G_x)\big)$ (with $x \in E$ and $u \in \overline{\mathbf{R}}$), defined by setting for every $z \in E$:

$$Ext_{-\infty}\big(\sigma_u(G_x)\big)(z) = \begin{cases} G(z-x)+u & \text{if } z \in S(G)_x, \\ -\infty & \text{otherwise.} \end{cases}$$

(In other words we translate G by x along E and shift it by u along $\overline{\mathbf{R}}$, and then extend it to the whole E by filling it with the grey-level $-\infty$). For every $W \in \mathcal{L}$, (35) implies that $\delta_G\dot{\delta}_G(W)$ is the supremum of all $Ext_{-\infty}\big(\sigma_u(G_x)\big)$ (with $x \in E$ and $u \in \overline{\mathbf{R}}$) which are below W. As $\dot{\delta}_G = \varepsilon_{\widetilde{G}}$, (18) and (36) imply that $\dot{\delta}_G\delta_G(W)$ is the infimum of all $Ext_{+\infty}\big(\sigma_{F(x)}(-G_x)\big)$ (with $x \in E$ and $u \in \overline{\mathbf{R}}$) which are above W.

In Figure 3 we illustrate the opening $\delta_G\dot{\delta}_G$ the case where $E = \mathbf{R}$ for a structuring function G shaped as a "pencil tip". Given a picture $F : \mathbf{R} \to \overline{\mathbf{R}}$, its transform $\delta_G\dot{\delta}_G(F)$ is the supremum of all "pencil tips" below F, and $\delta_G\dot{\delta}_G$ acts somewhat like a "pencil sharpener".

Given an arbitrary complete lattice $\mathcal{L}$, we can define some simple morphological openings and closings on it. We know (see Section IV) that the dual of the dilation $\langle \mathbf{O} \rangle$ is the erosion $\langle \mathbf{I} \rangle$. Thus $\langle \mathbf{O} \rangle = \langle \mathbf{O} \rangle \langle \mathbf{I} \rangle$ is a morphological opening and $\langle \mathbf{I} \rangle = \langle \mathbf{I} \rangle \langle \mathbf{O} \rangle$ is a morphological closing.

Let us recall the dyadic dilations and erosions described in (22) and (23); they were characterized by the fact that their result can take only two distinct values. Given $X, Y \in \mathcal{L}$ such that $Y \neq \mathbf{O}$ and $X \neq \mathbf{I}$, we defined $\delta_{X,Y}$ and $\varepsilon_{Y,X}$ by setting

$$\delta_{X,Y}(Z) = \begin{cases} \mathbf{O} & \text{if } Z \preceq X, \\ Y & \text{if } Z \not\preceq X, \end{cases}$$

$$\varepsilon_{Y,X}(Z) = \begin{cases} \mathbf{I} & \text{if } Z \succeq Y, \\ X & \text{if } Z \not\succeq Y, \end{cases}$$

for any $Z \in \mathcal{L}$. We can also define *dyadic* openings or closings, whose result can take only two distinct values; in view of Theorem 12, they take the following form: the opening α_Y (for $Y \neq \mathbf{O}$), and the closing φ_X (for $X \neq \mathbf{I}$), defined by

$$\begin{aligned} \alpha_Y(Z) &= \begin{cases} \mathbf{O} & \text{if } Z \not\succeq Y, \\ Y & \text{if } Z \succeq Y, \end{cases} \\ \varphi_X(Z) &= \begin{cases} \mathbf{I} & \text{if } Z \not\preceq X, \\ X & \text{if } Z \preceq X, \end{cases} \end{aligned} \qquad \text{for} \quad Z \in \mathcal{L}. \tag{38}$$

As we mentioned in Section IV, $\varepsilon_{Y,X} = \dot{\delta}_{X,Y}$. Now it is easy to see that for $X \neq \mathbf{I}$ and $Y \neq \mathbf{O}$ we have $\delta_{X,Y}\varepsilon_{Y,X} = \alpha_Y$ and $\varepsilon_{Y,X}\delta_{X,Y} = \varphi_X$. In other words, dyadic openings and closings are morphological openings and closings built with dyadic dilations and erosions.

Note that for $Y = \mathbf{O}$ and $X = \mathbf{I}$, the openings α_Y and φ_X reduce to the trivial morphological openings $\langle \mathbf{O} \rangle$ and $\langle \mathbf{I} \rangle$.

In the same way as dyadic dilations allowed us to prove Proposition 3, dyadic openings can be used to prove the following result, whose dual version involving closings is left to the reader:

Proposition 16 [Serra-Th]. *Every opening is a supremum of morphological openings.*

Proof. Let α be an opening and let $\mathcal{B}$ be the set of all its invariants in $\mathcal{L}$. For any $B \in \mathcal{B}$, δ_B is a morphological opening, because it is the trivial opening $\langle \mathbf{O} \rangle$ for $B = \mathbf{O}$, and a dyadic opening for $B \neq \mathbf{O}$. For any $X \in \mathcal{L}$ and $B \in \mathcal{B}$, we have $\alpha_B(X) = B$ if $B \preceq X$ and $\alpha_B(X) = \mathbf{O}$ if $B \not\preceq X$. Thus

$$[\bigvee_{B \in \mathcal{B}} \alpha_B](X) = \bigvee_{B \in \mathcal{B}} (\alpha_B(X)) = \bigvee \{B \mid B \in \mathcal{B}, B \preceq X\} = \alpha(X).$$

Therefore $\alpha = \bigvee_{B \in \mathcal{B}} \alpha_B$, and so α is a supremum of morphological openings. ∎

Although in practical applications the most used openings are morphological ones, sometimes other types of openings are needed. In the previous section we mentioned annular openings. We will now introduce a new type of opening which arose in the study of a particular problem in digital image processing: in [Ronse-conf] we investigated the possibility of a flexible generalization of the Min-Max filters of [Nakagawa-Rosenfeld] (that is, morphological openings by a flat structuring element), and proposed a combination of low rank and Max filters. The expression of our result in the framework of complete lattices was suggested by Serra:

Theorem 17. *Let δ^* be a dilation, let $\mathcal{F}$ be a set of dilations, and let $\dot{\mathcal{F}}$ be the set of erosions which are the morphological duals of elements of $\mathcal{F}$. Assume that for every $\delta \in \mathcal{F}$, $\delta \preceq \delta^*$, or dually (see Theorem 5 (iii)) that for every $\varepsilon \in \dot{\mathcal{F}}$, $\dot{\delta}^* \preceq \varepsilon$. Let $\eta_{\mathcal{F}} = \bigvee \dot{\mathcal{F}}$ and $\alpha_{\mathcal{F}} = 1 \wedge \delta^*\eta_{\mathcal{F}}$. Then $\eta_{\mathcal{F}}\alpha_{\mathcal{F}} = \eta_{\mathcal{F}}$ and $\alpha_{\mathcal{F}}$ is an opening. Moreover $\alpha_{\mathcal{F}} \succeq \delta^*\dot{\delta}^*$, and $\alpha_{\mathcal{F}} \succeq \delta\dot{\delta}$ for every dilation $\delta \in \mathcal{F}$.*

Proof. Let $\varepsilon \in \dot{\mathcal{F}}$. As $\varepsilon \succeq \dot{\delta}^*$, we have $\varepsilon\delta^*\eta_{\mathcal{F}} \succeq \dot{\delta}^*\delta^*\eta_{\mathcal{F}}$; as $\dot{\delta}^*\delta^*$ is extensive (see Theorem 14, dual version), we have $\dot{\delta}^*\delta^*\eta_{\mathcal{F}} \succeq \eta_{\mathcal{F}}$. Combining both inequalities, we get $\varepsilon\delta^*\eta_{\mathcal{F}} \succeq \eta_{\mathcal{F}}$, and so $\varepsilon \wedge \varepsilon\delta^*\eta_{\mathcal{F}} = \varepsilon$. As ε is an erosion, it commutes with $\bigwedge$, and hence:

$$\varepsilon[1 \wedge \delta^*\eta_{\mathcal{F}}] = \varepsilon \wedge \varepsilon\delta^*\eta_{\mathcal{F}} = \varepsilon.$$

As this holds for any $\varepsilon \in \dot{\mathcal{F}}$, we get:

$$\eta_{\mathcal{F}}\alpha_{\mathcal{F}} = [\bigvee \dot{\mathcal{F}}][1 \wedge \delta^*\eta_{\mathcal{F}}] = \bigvee_{\varepsilon \in \dot{\mathcal{F}}} \varepsilon[1 \wedge \delta^*\eta_{\mathcal{F}}] = \bigvee_{\varepsilon \in \dot{\mathcal{F}}} \varepsilon = \eta_{\mathcal{F}}.$$

We have shown that $\eta_{\mathcal{F}}\alpha_{\mathcal{F}} = \eta_{\mathcal{F}}$; it follows then that $\delta^*\eta_{\mathcal{F}}\alpha_{\mathcal{F}} = \delta^*\eta_{\mathcal{F}}$. Thus $\delta^*\eta_{\mathcal{F}}\alpha_{\mathcal{F}} \succeq 1 \wedge \delta^*\eta_{\mathcal{F}} = \alpha_{\mathcal{F}}$. As $\delta^*\eta_{\mathcal{F}}$ is increasing, this equality and Lemma 7 imply that $\alpha_{\mathcal{F}}$ is an opening.

For any $\varepsilon \in \dot{\mathcal{F}}$, we have $\varepsilon \succeq \dot{\delta}^*$, and so $\eta_{\mathcal{F}} = \bigvee \dot{\mathcal{F}} \succeq \dot{\delta}^*$. As δ^* is increasing, this implies that $\delta^*\eta_{\mathcal{F}} \succeq \delta^*\dot{\delta}^*$. Now $\delta^*\dot{\delta}^*$ is anti-extensive, and hence

$$\delta^*\dot{\delta}^* = 1 \wedge \delta^*\dot{\delta}^* \preceq 1 \wedge \delta^*\eta_{\mathcal{F}}.$$

For every dilation $\delta \in \mathcal{F}$, we have $\delta \preceq \delta^*$, and so $\delta\eta_{\mathcal{F}} \preceq \delta^*\eta_{\mathcal{F}}$. As $\dot{\delta} \preceq \eta_{\mathcal{F}}$ (by definition of $\eta_{\mathcal{F}}$) and δ is increasing, we have $\delta\dot{\delta} \preceq \delta\eta_{\mathcal{F}}$. Combining both inequalities, we get $\delta\dot{\delta} \preceq \delta^*\eta_{\mathcal{F}}$. Now $\delta\dot{\delta}$ is anti-extensive, and hence

$$\delta\dot{\delta} = 1 \wedge \delta\dot{\delta} \preceq 1 \wedge \delta^*\eta_{\mathcal{F}}. \blacksquare$$

It follows in particular (see Proposition 11) that the opening $\alpha_{\mathcal{F}}$ commutes with the two openings $\delta^*\dot{\delta}^*$ and $\delta\dot{\delta}$.

In [Matheron-FL], an increasing operator γ such that $\gamma = \gamma[1 \wedge \gamma]$ is called a $\wedge$-*over filter*. For any $\wedge$-over filter γ, we have $\gamma[1 \wedge \gamma] \succeq 1 \wedge \gamma$, and so Lemma 7 implies that $1 \wedge \gamma$ is an opening. Now Theorem 17 says in particular that $\delta^*\eta_{\mathcal{F}}$ is a $\wedge$-over filter.

In order to understand the behavior of this opening $\alpha_{\mathcal{F}}$, we must describe its invariants:

Proposition 18. *Let δ^*, $\mathcal{F}$, $\eta_{\mathcal{F}}$, and $\alpha_{\mathcal{F}}$ be as in Theorem 17. For any $B \in \mathcal{L}$, B is an invariant of $\alpha_{\mathcal{F}}$ iff to every $\delta \in \mathcal{F}$ we can associate some $X_\delta \in \mathcal{L}$ in such a way that*

$$\bigvee_{\delta\in\mathcal{F}} \delta(X_\delta) \preceq B \preceq \delta^*(\bigvee_{\delta\in\mathcal{F}} X_\delta) = \bigvee_{\delta\in\mathcal{F}} \delta^*(X_\delta). \tag{39}$$

Proof. (1°) Suppose first that B is an invariant of $\alpha_{\mathcal{F}} = 1 \wedge \delta^*\eta_{\mathcal{F}}$. For each $\delta \in \mathcal{F}$, set $X_\delta = \dot{\delta}(B)$. By the definition of morphological duality (see (25)), we have thus $\delta(X_\delta) \preceq B$, and so

$$\bigvee_{\delta\in\mathcal{F}} \delta(X_\delta) \preceq B.$$

As B is an invariant of $1 \wedge \delta^*\eta_{\mathcal{F}}$, we have $B \preceq \delta^*\eta_{\mathcal{F}}(B)$, and so

$$B \preceq \delta^*\eta_{\mathcal{F}}(B) = \delta^*[\bigvee_{\delta\in\mathcal{F}} \dot{\delta}](B) = \delta^*(\bigvee_{\delta\in\mathcal{F}} \dot{\delta}(B)) = \delta^*(\bigvee_{\delta\in\mathcal{F}} X_\delta).$$

Then (39) holds.

(2°) Suppose now that (39) holds. For every $\delta \in \mathcal{F}$, by (39) we have $\delta(X_\delta) \preceq B$. By the definition of morphological duality (see (25)), this implies that $X_\delta \preceq \dot{\delta}(B)$. Hence

$$\bigvee_{\delta\in\mathcal{F}} X_\delta \preceq \bigvee_{\delta\in\mathcal{F}} \dot{\delta}(B) = [\bigvee_{\delta\in\mathcal{F}} \dot{\delta}](B) = \eta_{\mathcal{F}}(B).$$

It follows that

$$\delta^*(\bigvee_{\delta\in\mathcal{F}} X_\delta) \preceq \delta^*\eta_{\mathcal{F}}(B),$$

and so by (39) we have $B \preceq \delta^*\eta_{\mathcal{F}}(B)$. Then B is an invariant of $1 \wedge \delta^*\eta_{\mathcal{F}} = \alpha_{\mathcal{F}}$. ∎

In order to understand what (39) means in practice, consider the case of binary or grey-level pictures on a space E. Here $\bigwedge$ distributes $\bigvee$ (as multiplication distributes addition). Therefore by setting $B_\delta = B \wedge \delta^*(X_\delta)$ for every $\delta \in \mathcal{F}$, equation (39) becomes:

$$\begin{aligned} B &= \bigvee_{\delta\in\mathcal{F}} B_\delta, \\ \text{where} \qquad \delta(X_\delta) &\preceq B_\delta \preceq \delta^*(X_\delta) \qquad \text{for every} \quad \delta \in \mathcal{F}. \end{aligned} \tag{40}$$

Thus the dual Moore family of invariants of $\alpha_{\mathcal{F}}$ is generated by pictures B_δ which are between invariants $\delta(X_\delta)$ of $\delta \in \mathcal{F}$ and corresponding invariants $\delta^*(X_\delta)$ of δ^*.

Let us illustrate this in the case of binary pictures on E. If δ^* is the dilation by a structuring element B^*, and $\mathcal{F}$ is the set of dilations by structuring elements in a set $\mathcal{P}$ of parts of B^*, then for $W \subseteq E$, its transform $\alpha_{\mathcal{F}}(W)$ is the union of all $X \subseteq E$ such that $B_x \subseteq X \subseteq B^*_x$ for some $B \in \mathcal{P}$ and $x \in E$. For example, if $\mathcal{P}$ is the set of all parts of B^* comprising a proportion λ of it $(0 < \lambda < 1)$, then $\alpha_{\mathcal{F}}$ will delete from W all portions of it which are too small to contain at least a proportion λ of a translate of B^*. This opening is subtler than the morphological opening by B^*, which deletes from W all portions of it which are too small to contain the whole of a translate of B^*.

In [Ronse-conf] this idea has been generalized to grey-level pictures for dilations by flat structuring functions (see (20) and (21)), which were called "Max filters". We proposed there an opening of the form $1 \wedge Max \cdot R_k$, where Max is a "Max filter" and R_k is the "k-th rank filter". As we explained there, this operator is useful for the extraction of narrow ridge features in a grey-level picture, and it is less sensitive to noisy bottoms than a composition of a "Min filter" followed by a "Max filter" (*i.e.*, a morphological opening).

In Figure 4 we illustrate this opening in the case of grey-level pictures with δ^* being the dilation by a structuring function G^* shaped as a "pencil tip" (see Figures 1 and 3), and $\mathcal{F}$ is a set of dilations by large portions of G^*. Then the opening $\alpha_{\mathcal{F}}$ will preserve all peaks containing a significant portion of a "pencil tip".

Let us note however that (40) is equivalent to (39) only in the case where $\bigwedge$ distributes $\bigvee$ (in particular for pictures of the form $E \to D$, where $D \subseteq \overline{\mathbf{R}}$, for example grey-level Euclidean pictures). Thus in the general case, we cannot use formula (40), but only (39).

An interesting property of the opening $\alpha_{\mathcal{F}}$ is that we can vary the set $\mathcal{F}$. When the dilations $\delta \in \mathcal{F}$ (which are all below δ^*) increase, $\eta_{\mathcal{F}}$ and $\alpha_{\mathcal{F}}$ decrease, and when $\mathcal{F} = \{\delta^*\}$, we have $\eta_{\mathcal{F}} = \dot{\delta}^*$ and $\alpha_{\mathcal{F}} = \delta^* \dot{\delta}^*$. Thus by making the elements of $\mathcal{F}$ tend together to δ^*, one makes $\alpha_{\mathcal{F}}$ tend to $\delta^* \dot{\delta}^*$.

VII. Interest of the abstract approach

The usefulness of morphological operators in image processing has been demonstrated throughout [Serra-IAMM]. Many transformations on images (for example thinning, thickening, skeletonization, convex hull, median filtering, connected component labelling) can be expressed in terms of basic morphological operations. Moreover, mathematical morphology has been applied for solving practical problems in various disciplines such as mineralogy, cytology, etc..

Besides well-known morphological operators (dilations, erosions, morphological openings and closings), the new type of opening defined in Theorem 17 seems to have many possible applications. We have made some first experiments on grey-level pictures representing X-ray images of coronary arteries; using this type of opening (with flat structuring functions), we obtain a flexible method for detecting blood vessels with variable trade-offs between detail preservation and noise suppression.

However one can wonder whether it is necessary to study morphological operators on this abstract level of complete lattices instead of restricting oneself to structuring functions for grey-level images and structuring elements for binary ones.

A first answer is that structuring functions are defined for grey-level pictures $\mathbf{R}^n \to \overline{\mathbf{R}}$ (or eventually $\mathbf{Z}^n \to \overline{\mathbf{Z}}$, where $\mathbf{Z}$ is the set of relative integers). As we mentioned in

Section III, if we consider pictures with grey-levels within a bounded interval $[a, b]$, then we must make several modifications to our computations (to eliminate out of bounds grey-levels), and a structuring function G must satisfy the condition $\sup_{x \in S(G)} G(x) = 0$. This condition did not arise from a study of structuring functions, but from lattice-theoretic considerations. Indeed, for a dilation δ, we must have $\delta(\mathbf{O}) = \mathbf{O}$ because $\mathbf{O} = \bigvee \emptyset$. Applying this condition to formula (17) with $F = \mathbf{O}$, where $\mathbf{O}$ is a picture with constant grey-level $a > -\infty$, we obtain precisely this condition on G.

Without doubt, the adaptation of structuring functions to bounded spaces requires the truncation of structuring functions along the borders, and here also lattice-theoretic consideration will be useful for checking whether our operations remain dilations, erosions, morphological openings, etc..

Moreover, we do not exclude the possibility of devising other morphological operators on grey-level images, which are not translation-invariant. For example one can use different structuring functions at various places in the image, because one can look for different features at different places. Therefore, even if one restricts oneself to grey-level images on a subset of a Euclidean space, lattice-theoretic considerations remain necessary.

Finally, we feel that the formalism of complete lattices allows us to express properties of morphological operations in a clear and concise way. On the other hand, details of more complicated formulas used in particular cases (for example (17), (19), (32), etc. in the case of structuring functions) tend to hide the underlying fundamental ideas.

One can nevertheless still ask why we did not restrict the complete lattice $\mathcal{L}$ to being the set of grey-level images $E \to D$, where $E \subseteq \mathbf{R}^n$ and $D \subseteq \overline{\mathbf{R}}$. This question is a valid one, and we could indeed have made such a restriction. However our choice not to make any particular assumption on the nature of the complete lattice $\mathcal{L}$ leads only to one single conceptual abstraction, and it does not complicate anything in the exposition of properties of morphologic operators. Moreover, this formalism can help us to distinguish between properties of images which are due to the structure of complete lattice and those which are due to particular geometric and arithmetic properties of the set of maps $E \to \overline{\mathbf{R}}$ for $E \subseteq \mathbf{R}^n$. We also do not exclude the possibility of applying results concerning $\mathcal{L}$ to other complete lattices derived from it, for example the set $\mathcal{O}$ of operators or the set $\mathcal{I}$ of increasing operators.

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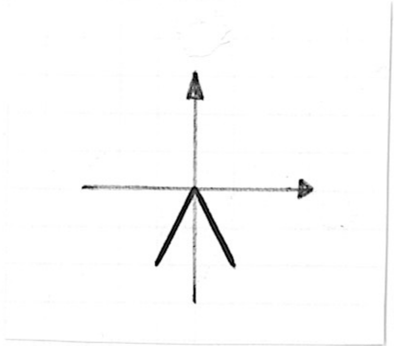
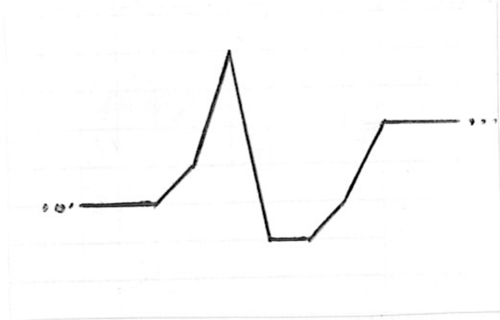
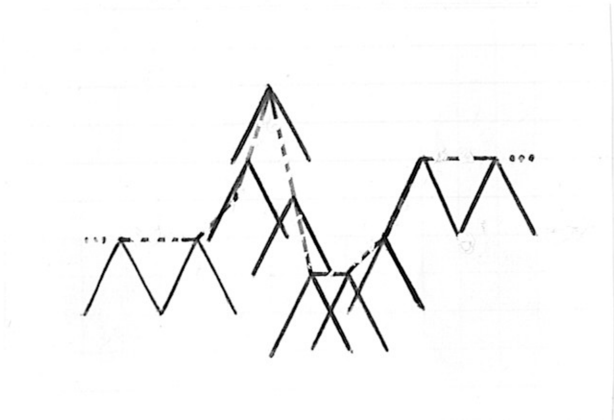


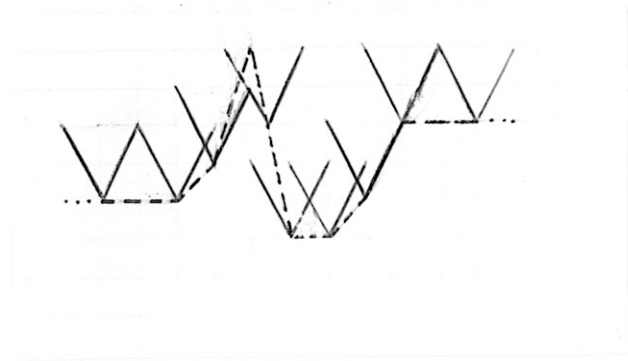
The structuring function G.



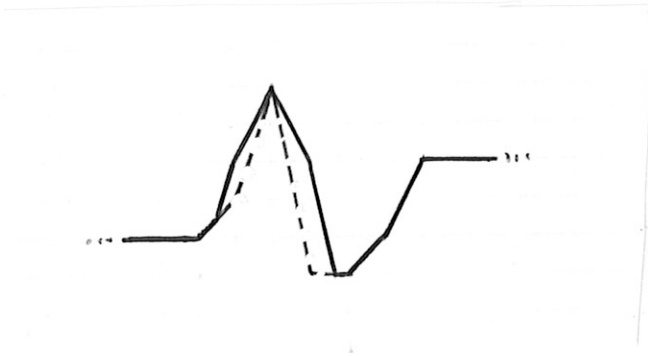
The picture $F : \mathbb{R} \to \overline{\mathbb{R}}$.



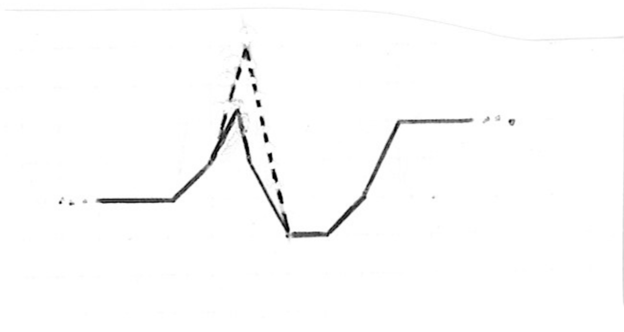
$Ext_{-\infty}\big(\sigma_{F(x)}(G_x)\big)$, $x \in \mathbb{R}$.



$Ext_{+\infty}\big(\sigma_{F(x)}(-G_x)\big)$, $x \in \mathbb{R}$.

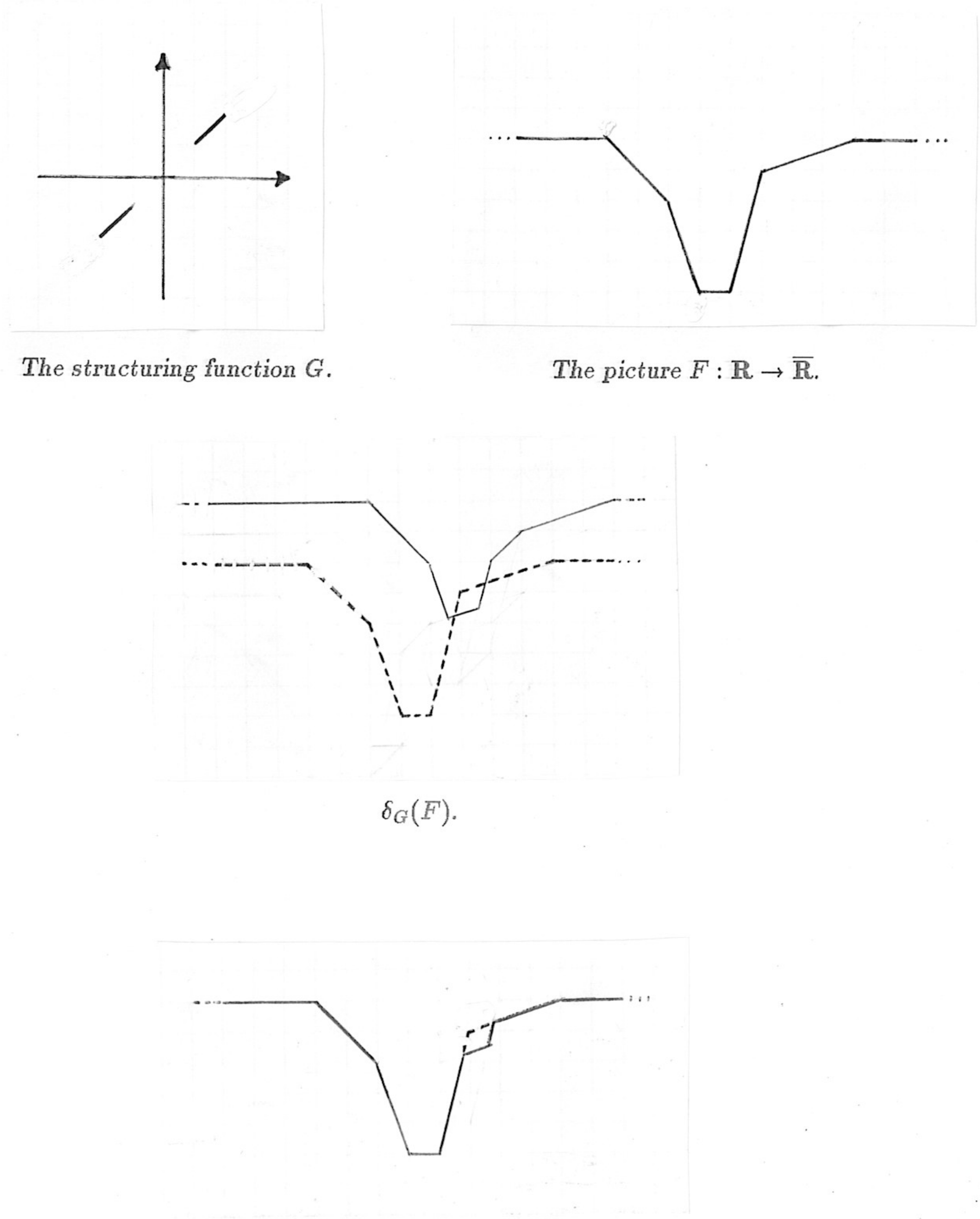


$\delta_G(F)$.



$\varepsilon_G(F)$.

Figure 1. *The dilation and erosion of picture F by a structuring function G.*



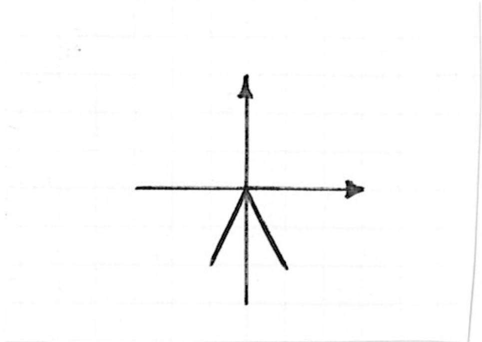
The structuring function G.

The picture $F : \mathbb{R} \to \overline{\mathbb{R}}$.

$\delta_G(F)$.

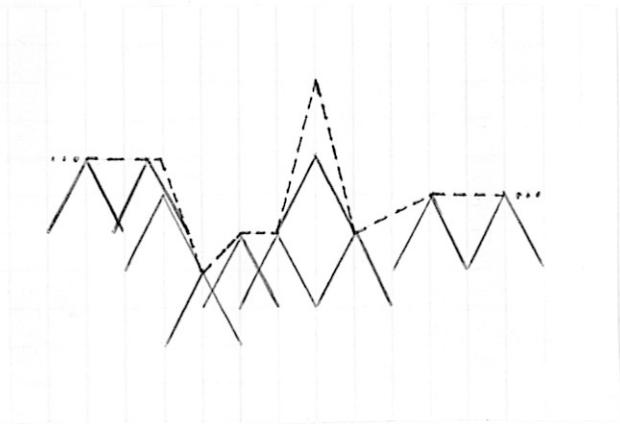
$[1 \wedge \delta_G](F)$.

Figure 2. *The annular opening $1 \wedge \delta_G$ applied to picture F.*

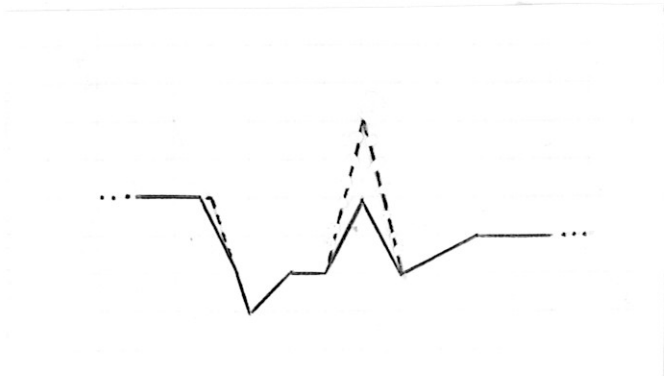


The structuring function G.

The picture $F : \mathbb{R} \to \overline{\mathbb{R}}$.

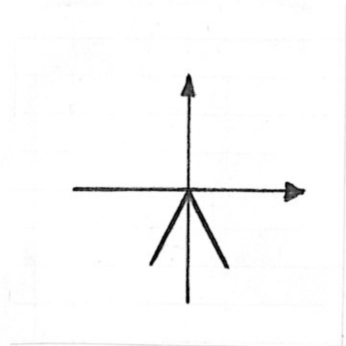


$Ext_{-\infty}\big(\sigma_u(G_x)\big) \preceq F,\ x \in \mathbb{R}.$

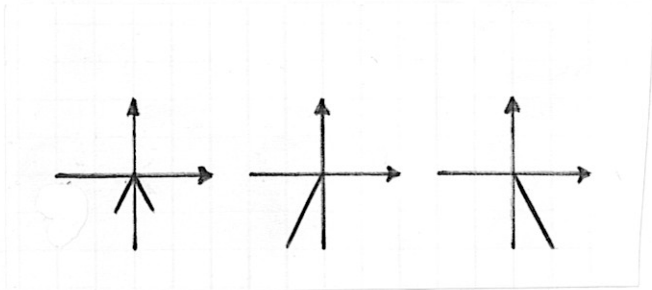


$\delta_G \dot{\delta}_G(F).$

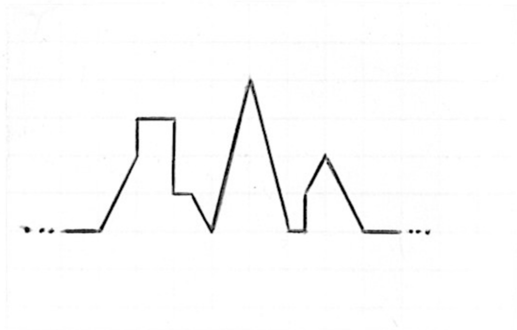
Figure 3. *The morphological opening $\delta_G \dot{\delta}_G$ applied to picture F.*



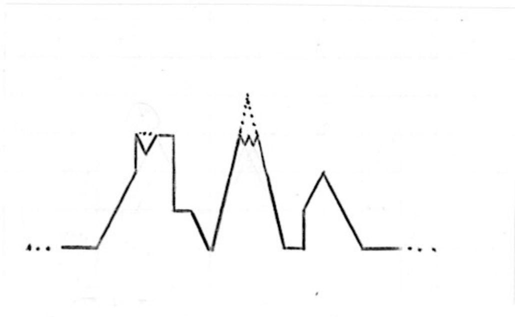
The structuring function G^*.



The structuring functions G_1, G_2, G_3.



The picture $F : \mathbb{R} \to \overline{\mathbb{R}}$.



$[1 \wedge \delta_{G^*}[\dot{\delta}_{G_1} \vee \dot{\delta}_{G_2} \vee \dot{\delta}_{G_3}]](F)$.

Figure 4. *The opening* $1 \wedge \delta_{G^*}[\dot{\delta}_{G_1} \vee \dot{\delta}_{G_2} \vee \dot{\delta}_{G_3}]$ *applied to picture* F.