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On the continuity of local finitary transformations of complete lattices

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Abstract. We show that a local finitary operator on an up-distributive complete lattice is lower semi-continuous. When the lattice satisfies some additional constraints, such an operator is continuous. This generalizes previous results by Heijmans, Serra, and the author. It applies in particular to local neighbourhood operations in binary, grey-level, or colour digital images.

Key words.

Résumé.

Mots clés.

1. Introduction

In [H1,H2,HS,RH,R,S], various results have been given concerning the continuity of operators on complete lattices. They were then applied to image processing. In this note we generalize some of the most recent results from [HS,R]. We must beforehand introduce our terminology.

Definitions and notation. Let $(\mathcal{L}, \leq)$ be a *complete* lattice. This means that $\mathcal{L}$ is partially ordered by $\leq$, and every subset S of $\mathcal{L}$ has a supremum $\bigvee S$ and an infimum $\bigwedge S$. Let X_n, $n \in \mathbb{N}$, be a sequence in $\mathcal{L}$. We say that $X_n \uparrow X$ (pronounce: X_n *up-converges* to X) if this sequence is increasing (*i.e.*, $X_n \leq X_{n+1}$ for every $n \in \mathbb{N}$), and $X = \bigvee_{n\in\mathbb{N}} X_n$. Given $Y \in \mathcal{L}$ and an increasing sequence X_n, $n \in \mathbb{N}$, with $X_n \uparrow X$, we have

$$X \vee Y = \Big(\bigvee_{n\in\mathbb{N}} X_n\Big) \vee Y = \bigvee_{n\in\mathbb{N}} (X_n \vee Y),$$

in other words $X_n \vee Y \uparrow X \vee Y$. We say that $\mathcal{L}$ is *up-distributive* if whenever $X_n \uparrow X$, then for any $Y \in \mathcal{L}$ we have

$$X \wedge Y = \bigvee_{n\in\mathbb{N}} (X_n \wedge Y),$$

in other words $X_n \wedge Y \uparrow X \wedge Y$. Note that up-distributivity is independent from ordinary distributivity (namely, that $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ for all $A, B, C \in \mathcal{L}$). For example, in $\mathbb{R}^d$ the complete lattice of convex sets is up-distributive but not distributive, while the one of closed sets is distributive but not up-distributive.

We say that $\mathcal{L}$ is *σ-sup-distributive* if

$$\Big(\bigvee_{n\in\mathbb{N}} X_n\Big) \wedge Y = \bigvee_{n\in\mathbb{N}} (X_n \wedge Y)$$

for any $Y \in \mathcal{L}$ and any sequence X_n ($n \in \mathbb{N}$) in $\mathcal{L}$ (not only increasing ones). Given a sequence X_n, if we define $Y_m = \bigvee_{n=0}^{m} X_n$ for all $m \in \mathbb{N}$, then $Y_m \uparrow \bigvee_{n\in\mathbb{N}} X_n$; it is thus easily shown that $\mathcal{L}$ is σ-sup-distributive if and only if it is both distributive and up-distributive.

We define in an analogous way dual concepts, namely "$X_n \downarrow X$" (pronounce: X_n *down-converges* to X), *down-distributivity*, and *σ-inf-distributivity*.

Let X_n, $n \in \mathbb{N}$, be a sequence in $\mathcal{L}$. We define

$$\liminf_{n\in\mathbb{N}} X_n = \bigvee_{n\in\mathbb{N}} \bigwedge_{m\geq n} X_m;$$
$$\limsup_{n\in\mathbb{N}} X_n = \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n} X_m.$$

Obviously $\liminf X_n \leq \limsup X_n$; we say that X_n *converges* to X, and write $X_n \to X$ or $X = \lim X_n$, if $X = \liminf X_n = \limsup X_n$.

A function $\mathcal{L} \to \mathcal{L}$ will be called an *operator*. An operator θ is said *lower semi-continuous* if for a sequence X_n converging to X we have $\theta(X) \leq \liminf \theta(X_n)$. Dually, we say that θ is *upper semi-continuous* if for $X_n \to X$ we have $\limsup \theta(X_n) \leq \theta(X)$. When θ is both lower and upper semi-continuous, then we say that it is *continuous*; this means that for $X_n \to X$ we have $\theta(X_n) \to \theta(X)$. Note that Heijmans and Serra [HS] use the expressions "↑-continuous" and "↓-continuous" where we say "lower semi-continuous" and "upper semi-continuous".

An element F of $\mathcal{L}$ is called *finite* if the set $\{G \in \mathcal{L} \mid G \leq F\}$ is finite. For every $X \in \mathcal{L}$, write $\text{fin}(X)$ for the set of all $F \in \mathcal{L}$ such that $F \leq X$ and F is finite. We say that an operator $\theta : \mathcal{L} \to \mathcal{L}$ is *finitary* if for any $X \in \mathcal{L}$ we have

$$\theta(X) \leq \bigvee_{F \in \text{fin}(X)} \theta(F). \tag{1}$$

A subset ℓ of a complete lattice $\mathcal{L}$ is called *sup-generating* if every $X \in \mathcal{L}$ is the supremum of a subset of ℓ, in other words $X = \sup \ell(X)$, where $\ell(X) = \{x \in \ell \mid x \leq X\}$. Note that X is finite if and only if the set $\ell(X)$ is finite.

Let the complete lattice $\mathcal{L}$ contain a sup-generating family ℓ. An increasing operator $\theta : \mathcal{L} \to \mathcal{L}$ is called *ℓ-finitary* if to every $h \in \ell$ we can associate a finite $M(h) \in \mathcal{L}$ in such a way that for every $X \in \mathcal{L}$ we have

$$h \leq \theta(X) \quad \text{if and only if} \quad h \leq \theta(X \wedge M(h)). \tag{2}$$

The map $M : \ell \to \mathcal{L} : h \mapsto M(h)$ is called the *mask function*. Note that an ℓ-finitary operator is finitary, because for any $X \in \mathcal{L}$ we have

$$\theta(X) \leq \bigvee_{h \in \ell} \theta(X \wedge M(h)),$$

In Section 4 of [R] we considered several types of semi-continuity properties of operators, and obtained a few results (which will be discussed in more detail in the next section). In particular, let us mention the following ones:

— *If $\mathcal{L}$ is up-distributive, every increasing and finitary operator is lower semi-continuous* (see [R], Theorem 4.4 and after).

— *If $\mathcal{L}$ is up-distributive and has a sup-generating family ℓ, every increasing and ℓ-finitary operator is continuous* (see [R], after Theorem 4.5).

Heijmans and Serra showed on Proposition 6.2 of [HS] that *for $\mathcal{L} = \mathcal{P}(\mathcal{E})$* and ℓ consisting of all singletons, *every ℓ-finitary operator is continuous*. We will generalize the latter two results by showing that when $\mathcal{L}$ satisfies certain properties, every ℓ-finitary operator is continuous.

2. New Theorems

We recall this elementary result from [R]:

Lemma 1. *Assume that $\mathcal{L}$ is up-distributive. Take $X, F \in \mathcal{L}$, where F is finite, and let X_n, $n \in \mathbb{N}$, be a sequence in $\mathcal{L}$ such that $X_n \downarrow X$ or $X_n \uparrow X$. Then there is some $m \in \mathbb{N}$ such that for any $n \geq m$, $X_n \wedge F = X \wedge F$.*

In order to show our first result, we require beforehand a generalization of Lemma 1:

Lemma 2. *Assume that $\mathcal{L}$ is up-distributive. Let $X, F \in \mathcal{L}$, where F is finite, and let X_n, $n \in \mathbb{N}$, be a sequence in $\mathcal{L}$ such that $X_n \to X$. Then there is some $m \in \mathbb{N}$ such that for any $n \geq m$, $X_n \wedge F = X \wedge F$.*

Proof. We have $X = \limsup X_n = \liminf X_n$. For each $n \in \mathbb{N}$ we define

$$Y_n = \bigwedge_{m \geq n} X_m \qquad \text{and} \qquad Z_n = \bigvee_{m \geq n} X_m.$$

Then

$$Y_n \uparrow \bigvee_{n\in\mathbb{N}} \bigwedge_{m\geq n} X_m = \liminf X_n = X \qquad \text{and} \qquad Z_n \downarrow \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n} X_m = \limsup X_n = X.$$

As $\mathcal{L}$ is up-distributive, by Lemma 1 there exist $m_0, m_1 \in \mathbb{N}$ such that for $n \geq m_0$ we have $Y_n \wedge F = X \wedge F$, and for $n \geq m_1$ we have $Z_n \wedge F = X \wedge F$. Let $m = \max(m_0, m_1)$; then for $n \geq m$ we have $Y_n \wedge F = Z_n \wedge F = X \wedge F$; now $Y_n \leq X_n \leq Z_n$, so that we get $Y_n \wedge F \leq X_n \wedge F \leq Z_n \wedge F$; hence $X_n \wedge F = X \wedge F$ for $n \geq m$. ∎

Theorem 3. *Let $\mathcal{L}$ be up-distributive and have a sup-generating family ℓ, and let θ be an ℓ-finitary operator. Then θ is lower semi-continuous.*

Proof. Let $X_n \to X$; take $h \in \ell$ such that $h \leq \theta(X)$. Then $h \leq \theta(X \wedge M(h))$ (since θ is ℓ-finitary). By Lemma 2 there is some $m \in \mathbb{N}$ such that for every $n \geq m$ we have $X_n \wedge M(h) = X \wedge M(h)$; thus for $n \geq m$ we have $h \leq \theta(X_n \wedge M(h))$, so that $h \leq \theta(X_n)$ (since θ is ℓ-finitary). Hence $h \leq \bigwedge_{n\geq m} \theta(X_n)$, so that

$$h \leq \bigvee_{m\in\mathbb{N}} \bigwedge_{n\geq m} \theta(X_n) = \liminf \theta(X_n).$$

As $\theta(X)$ is sup-generated by all such h below it, we deduce that $\theta(X) \leq \liminf \theta(X_n)$, and θ is lower semi-continuous. ∎

Note that the assumption that $\mathcal{L}$ is up-distributive cannot be dropped. Indeed if $\mathcal{L}$ is not up-distributive, then there exists an increasing sequence $X_n \in \mathcal{L}$ ($n \in \mathbb{N}$) and $X, Y \in \mathcal{L}$ such that $X_n \uparrow X$ but $X \wedge Y \neq \bigvee_{n\in\mathbb{N}}(X_n \wedge Y)$; as $X_n \wedge Y \leq X \wedge Y$ for each n, we get thus $X \wedge Y > \bigvee_{n\in\mathbb{N}}(X_n \wedge Y)$. Let $Z = \bigvee_{n\in\mathbb{N}}(X_n \wedge Y)$. As $X_n \uparrow X$ and $X_n \wedge Y \uparrow Z$, we have $X_n \to X$ and $X_n \wedge Y \to Z$ (see [HS], Proposition 2.3). Hence

$$\lim(X_n \wedge Y) = Z < X \wedge Y = (\lim X_n) \wedge Y.$$

This implies that the increasing operator $\psi : \mathcal{L} \to \mathcal{L} : A \mapsto A \wedge Y$ is not lower semi-continuous.

We might wonder whether we can show that θ is also upper semi-continuous. Following [R], we say that θ is *down-continuous* if for every decreasing sequence X_n, $n \in \mathbb{N}$, with $X_n \downarrow X$, we have

$$\theta(X) \geq \bigwedge_{n\in\mathbb{N}} \theta(X_n). \tag{3}$$

As explained in [R], upper semi-continuity implies down-continuity, and for an increasing operator, they are equivalent. We showed in Theorem 4.5 of [R] that:

— *If $\mathcal{L}$ has a sup-generating family ℓ, every ℓ-finitary operator θ is down-continuous.*

It follows in particular that if θ is also increasing, then it will be upper semi-continuous. When θ is not increasing, we can show the upper semi-continuity of θ by imposing additional constraints on $\mathcal{L}$; recall that for every $X \in \mathcal{L}$, $\ell(X) = \{x \in \ell \mid x \leq X\}$.

Theorem 4. *Let $\mathcal{L}$ be σ-sup-distributive and have a sup-generating family ℓ; let θ be an ℓ-finitary operator with mask function M such that for any $h \in \ell$, $\bigvee_{k\in\ell(h)} M(k)$ is finite. Then θ is continuous.*

Proof. Let $X_n \to X$; take $h \in \ell$ such that $h \leq \limsup \theta(X_n)$. Now $\limsup \theta(X_n) = \bigwedge_{m\in\mathbb{N}} \bigvee_{n\geq m} \theta(X_n)$, so that for every $m \in \mathbb{N}$, $h \leq \bigvee_{n\geq m} \theta(X_n)$. Let $N(h) = \bigvee_{k\in\ell(h)} M(k)$; as $N(h)$ is finite, there exists by Lemma 2 some $m \in \mathbb{N}$ such that $X_n \wedge N(h) = X \wedge N(h)$ for all $n \geq m$. As $h \leq \bigvee_{n \geq m} \theta(X_n)$, from the σ-sup-distributivity of $\mathcal{L}$ we deduce that

$$h = \bigvee_{n \geq m} (h \wedge \theta(X_n)). \tag{4}$$

Take any $n \geq m$ and $k \in \ell$ such that $k \leq h \wedge \theta(X_n)$; the fact that θ is ℓ-finitary implies that $k \leq \theta(X_n \wedge M(k))$; as $n \geq m$, $X_n \wedge N(h) = X \wedge N(h)$, and as $k \in \ell(h)$, $M(k) \leq N(h)$, so that we have $X_n \wedge M(k) = X_n \wedge N(h) \wedge M(k) = X \wedge N(h) \wedge M(k) = X \wedge M(k)$, and hence $k \leq \theta(X \wedge M(k))$; thus $k \leq \theta(X)$ since θ is ℓ-finitary. Now $h \wedge \theta(X_n)$ is sup-generated by all $k \in \ell$ below it, and hence we get $h \wedge \theta(X_n) \leq \theta(X)$ for $n \geq m$. By (4) it follows that $h \leq \theta(X)$. As $\limsup \theta(X_n)$ is sup-generated by all such $h \in \ell$ below it, we deduce that $\limsup \theta(X_n) \leq \theta(X)$. Therefore θ is upper semi-continuous. As θ is lower semi-continuous by Theorem 3, it will be continuous. ∎

Note that in a distributive lattice, the supremum of two finite elements is finite. Indeed, for F, F' finite and $G \leq F \vee F'$, we have $G = (G \wedge F) \vee (G \wedge F')$; as $G \wedge F \leq F$ and $G \wedge F' \leq F'$, there is only a finite number of possibilities for each of them, so that we have a finite numbre of choices for G. By induction, the supremum of a finite number of finite elements is finite. It follows that the assumption that we made on the mask function M is satisfied in the following situations:

— all elements of ℓ are finite;

— for $h, k \in \ell$, $k \leq h$ implies $M(k) \leq M(h)$;

— for any $h \in \ell$, there is only a finite number of $k \in \ell(h)$ such that $M(k) \not\leq M(h)$.

Note that the first two situations are particular cases of the third one.

We do not know whether the condition on the mask function can be weakened. On the other hand, the σ-sup-distributivity condition on $\mathcal{L}$ cannot. Recall that it means that $\mathcal{L}$ is both distributive and up-distributive. The following two counterexample show that neither of the two partial conditions can be dropped, even when ℓ consists of points:

Example 1. A subset X of $\mathbb{Z}$ is *convex* if it is the intersection of $\mathbb{Z}$ and a convex subset of $\mathbb{R}$, in other words if for any $a, b \in X$ with $a < b$, X contains all $m \in \mathbb{Z}$ such that $a < m < b$. Let $\mathcal{L}$ be the set of all convex subsets of $\mathbb{Z}$, ordered by inclusion. As an arbitrary intersection of convex sets and the union of an increasing sequence of convex sets are convex, $\mathcal{L}$ is an *up-distributive complete lattice*. Now define θ as follows:

$$\theta(X) = \begin{cases} \{-x\} & \text{if } X = \{x\}, \text{ where } x > 0 \text{ and } x \text{ is even;} \\ X & \text{otherwise.} \end{cases}$$

As X is convex, $X = \{x\}$ if and only if $x \in X$ and $x - 1, x + 1 \notin X$. Thus for every point $p \in \mathbb{Z}$, whether $p \in \theta(X)$ depends only on the intersection of X with $M(p)$, where we have:

$$M(p) = \begin{cases} \{p\} & \text{if } p \text{ is odd or } p = 0; \\ \{p-1, p, p+1\} & \text{if } p \text{ is even and } p > 0; \\ \{p, -p-1, -p, -p+1\} & \text{if } p \text{ is even and } p < 0. \end{cases}$$

Hence θ is *ℓ-finitary* for ℓ being the set of singletons. Now define $X_n = \{n\}$ for all $n \in \mathbb{N}$. Then $X_n \to \emptyset$, while $\limsup \theta(X_n) = \mathbb{Z}$, because for each $m \in \mathbb{N}$, $\bigvee_{n \geq m} \theta(X_n)$ is convex and contains arbitrary large positive and negative numbers. Thus θ is *not upper semi-continuous*.

EXAMPLE 2. Let $\mathcal{L}$ be the set of topologically closed subsets of $\mathbb{R}$, ordered by inclusion. As an arbitrary intersection of closed sets and a finite union of closed sets are closed, $\mathcal{L}$ is a *distributive complete lattice*. Let $X_n = [2^{-n}, 1]$ for each $n \in \mathbb{N}$, and define $\theta : \mathcal{L} \to \mathcal{L}$ by $\theta(Z) = Z^c \cap \{0\}$, where Z^c is the complement of Z in $\mathbb{R}$. Then $X_n \to X = [0, 1]$, with $\theta(X) = \emptyset$, while $\theta(X_n) = \{0\}$ for each $n \in \mathbb{N}$, so that $\theta(X_n) \to \{0\}$. Thus $\theta(\lim X_n)$ is strictly smaller than $\lim \theta(X_n)$, and so θ is *not upper semi-continuous*. However θ is *ℓ-finitary*, where ℓ is the set of singletons and we set $M(p) = \{0\}$ for all $p \in \mathbb{R}$; indeed

$$\theta(Z) = Z^c \cap \{0\} = (Z^c \cup \{0\}^c) \cap \{0\} = (Z \cap \{0\})^c \cap \{0\} = \theta(Z \cap \{0\}) = \theta(Z \cap M(p))$$

for all $p \in \mathbb{R}$.

3. APPLICATION TO DIGITAL IMAGE PROCESSING

An important application of Theorems 3 and 4 lies in the neighbourhood operations used in digital image processing. Let $\mathcal{E}$ be a set, for instance a subset of the digital space $\mathbb{Z}^d$ for some $d \geq 1$. We consider binary images on $\mathcal{E}$ as subsets of $\mathcal{E}$, and grey-level images as functions $\mathcal{E} \to \mathcal{G}$, where $\mathcal{G}$ is the set of grey-levels. Let us assume that $\mathcal{G}$ is a *finite* lattice with least element g_0 and greatest element g_1. Thus $(\mathcal{P}(\mathcal{E}), \subseteq)$ is the complete lattice of binary images, while $(\mathcal{G}^{\mathcal{E}}, \leq)$ is the complete lattice of grey-level images.

For binary images, we take ℓ to be the set of singletons of $\mathcal{E}$, and the fact that θ is ℓ-finite means that to every $p \in \mathcal{E}$ one associates a finite window $W(p) \subseteq \mathcal{E}$ such that

$$\forall X \in \mathcal{P}(\mathcal{E}), p \in \mathcal{E}, \qquad p \in \theta(X) \quad \Longleftrightarrow \quad p \in \theta(X \cap W(p)). \tag{5}$$

In other words whether $\theta(X)$ contains p depends only on the configuration of points of X on $W(p)$. This condition, as expressed now, is independent from the ordering relation $\subseteq$. In [HS,R] it is shown that it is its own dual by complementation, in other words

$$\forall X \in \mathcal{P}(\mathcal{E}), p \in \mathcal{E}, \qquad p \in [\theta(X^c)]^c \quad \Longleftrightarrow \quad p \in \big[\theta\big([X \cap W(p)]^c\big)\big]^c.$$

This is nothing but (5) expressed for the dual operator θ^* defined by $\theta^*(X) = [\theta(X^c)]^c$.

As $\mathcal{P}(\mathcal{E})$ is σ-sup-distributive and singletons are finite, by Theorem 4 an operator θ satisfying (5) will be continuous. Alternately, by Theorem 3 both θ and θ^* will be lower semi-continuous, in other words θ will be continuous. This was first shown in Proposition 6.2 of [HS].

We refer to [H1] for a number of examples in the Boolean case; further developments can also be found in [H2,HS].

In the grey-level case, we take ℓ to be the set of all pulse functions $h = f_{p,g}$ $(g > g_0)$, having grey-level g on p and g_0 elsewhere. The condition that θ is ℓ-finitary means that to each such h we associate some grey-level function $M(h)$ with finite support (that is, $M(h)(p) > g_0$ only for a finite set of points p), so that for each $X \in \mathcal{L}$, $h \leq \theta(X)$ if and only if $h \leq \theta(X \wedge M(h))$. Here $M(h)$ can vary as p is fixed and g varies along $\mathcal{G}$. As it is expressed now, this condition is weaker than the requirement of *finitary local knowledge*, namely that the grey-level of $\theta(X)$ depends only on the configuration of grey-levels of X on the finite set $W(p)$. For $A \subseteq \mathcal{E}$, let χ_A be the characteristic function of A, having grey-level g_1 on A and g_0 elsewhere; for $p \in \mathcal{E}$, let $\chi_p = \chi_{\{p\}} = f_{p,g_1}$. Note that χ_A is finite for a finite set A; in particular χ_p and $\chi_{W(p)}$ are finite. Then we can express finitary local knowledge as follows:

$$\forall X \in \mathcal{G}^{\mathcal{E}}, p \in \mathcal{E}, \qquad \chi_p \wedge \theta(X) = \chi_p \wedge \theta(X \wedge \chi_{W(p)}). \tag{6}$$

Finitary local knowledge, as expressed above, is also independent of the ordering $\leq$ on $\mathcal{L}$. One can show that (6) leads to

$$\forall X \in \mathcal{G}^{\mathcal{E}}, p \in \mathcal{E}, \qquad \chi_{\{p\}^c} \vee \theta(X) = \chi_{\{p\}^c} \vee \theta(X \vee \chi_{W(p)^c}).$$

This is nothing but (6) expressed in the dual lattice (with the order $\leq$ reversed).

As $\mathcal{G}^{\mathcal{E}}$ is σ-sup-distributive and characteristic functions χ_p are finite, by Theorem 4 an operator θ satisfying (6) will be continuous. Alternately, by Theorem 3 θ will be lower semi-continuous for both $(\mathcal{G}^{\mathcal{E}}, \leq)$ and its dual $(\mathcal{G}^{\mathcal{E}}, \geq)$, in other words θ will be continuous.

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