Proof assistants for teaching proof and proving, motivations and challenges

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What is a proof assistant ?

A software which allows to :

- define mathematical concepts/programs
- mechanically check proofs of theorems/programs

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What it is not ?

- An automated theorem prover
- A tool that helps in finding the proofs

History of proof assistants

- ACL Boyer-Moore (75-)
- LCF Milner (72)
- Automath De Bruijn (67)
- Mizar Trybulec (73-)
- Isabelle Paulson (86-)
- Coq Huet-Coquand (84-) (ACM Software System Award 2013)
- ...
- Lean Moura (2013)

Languages to describe proofs

Imperative We give orders (called tactics) to complete the proof tree

Declarative The proof is a sequence of mathematical assertions and their justification.

```
example (p q r : Prop) : p ∧ (q ∨ r) ↔ (p ∧ q) ∨ (p ∧ r) :=
begin
  apply iff.intro,
    intro H,
    apply (or.elim (and.elim_right H)),
      intro Hq,
      apply or.intro_left,
      apply and.intro,
        exact (and.elim_left H),
      exact Hq,
    intro Hr,
    apply or.intro_right,
    apply and.intro,
    exact (and.elim_left H),
    exact Hr,
  intro H,
  apply (or.elim H),
    intro Hpq,
    apply and.intro,
      exact (and.elim_left Hpq),
    apply or.intro_left,
    exact (and.elim_right Hpq),
  intro Hpr,
  apply and.intro,
    exact (and.elim_left Hpr),
  apply or.intro_right,
  exact (and.elim_right Hpr)
end
```

```
theorem sqrt_prime_irrational:
  fixes p :: nat
  assumes "prime p"
  shows "sqrt p ∉ ℚ"
proof
  from ‹prime p› have p: "p > 1" by (rule prime_gt_1_nat)
  assume "sqrt p ∈ ℚ"
  then obtain m n :: nat
    where n: "n ≠ 0"
      and sqrt_rat: "¦sqrt p¦ = m / n"
      and "coprime m n" by (rule Rats_abs_nat_div_natE)
  have eq: "m^2 = p * n^2"
  proof -
    from n and sqrt_rat have "m = ¦sqrt p¦ * n" by simp
    then have "m^2 = (sqrt p)^2 * n^2" by (simp add: power_mult_distrib)
    also have "(sqrt p)^2 = p" by simp
    also have "... * n^2 = p * n^2" by simp
    finally show ?thesis by linarith
  qed
  have "p dvd m ∧ p dvd n"
  proof
    from eq have "p dvd m^2" ..
    with ‹prime p› show "p dvd m" by (rule prime_dvd_power)
    then obtain k where "m = p * k" ..
    with eq have "p * n^2 = p^2 * k^2" by algebra
    with p have "n^2 = p * k^2" by (simp add: power2_eq_square)
    then have "p dvd n^2" ..
    with ‹prime p› show "p dvd n" by (rule prime_dvd_power)
  qed
  then have "p dvd gcd m n" by simp
  with ‹coprime m n› have "p = 1" by simp
  with p show False by simp
qed
```

Some success of proof assistants

- CompCert (X. Leroy, ACM Award 2022)
- seL4 Micro Kernel (ACM Award 2023)
- 4 Color theorem 2007, Feit-Thompson 2012 (Gonthier)
- Flyspeck 2015 (Hales)
- Polynomial Freiman-Ruzsa 2023 (Tao)

Motivations for using a proof assistant ?

- immediate feedback : forbids non-sense statements, forbids incorrect reasoning, detect scope and freshness errors,. . .
- clarify role of statements : axiom, lemma, hypothesis, premise, conclusion, goal, . . .
- clarify theoretical status [1] (conjecture, consequence of hypotheses, sufficient condition to obtain the goal, . . .) and operational status : premise/conclusion
- clarify allowed logical rules
- impartiality
- micro-world : definitions and allowed theorems
- clarify what is a proof forbidding other types of argumentation
- *gamification*

1. Duval's terminology

Different potential goals

- To teach what is a proof
- To teach logic
- To teach software foundations
- To automate proof-checking
- To teach maths in general

Existing tools

Two communities :

1. Didactics of mathematics
2. Interactive theorem proving

ITP Community

Computer Science

- logic
- proof of programs, semantics, software foundations

U-Penn, Portland, Princeton, Harvard, Warsaw, CNAM, Lyon, Nice, Paris, Strasbourg, ...

Maths

- Bachelor - Maths : Nijmegen (ProofWeb), Nice (CoqWeb), Chambery (Phox), Paris (Lean), Strasbourg (Edukera/LeanVerbose/Deaduction)...
- ...

My experiments

M2 CS : Semantics, Hoare's logic, formalized in Coq.

M1 CS : Formalization of proofs in geometry.

M1 CS : Course about formal theorem with/about Coq.

L2 CS : Course about logic, Edukera in Logic mode (natural deduction)

L1 Maths and L1 CS : Introduction to proof

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L1 Maths and L1 CS : **Introduction to proof**

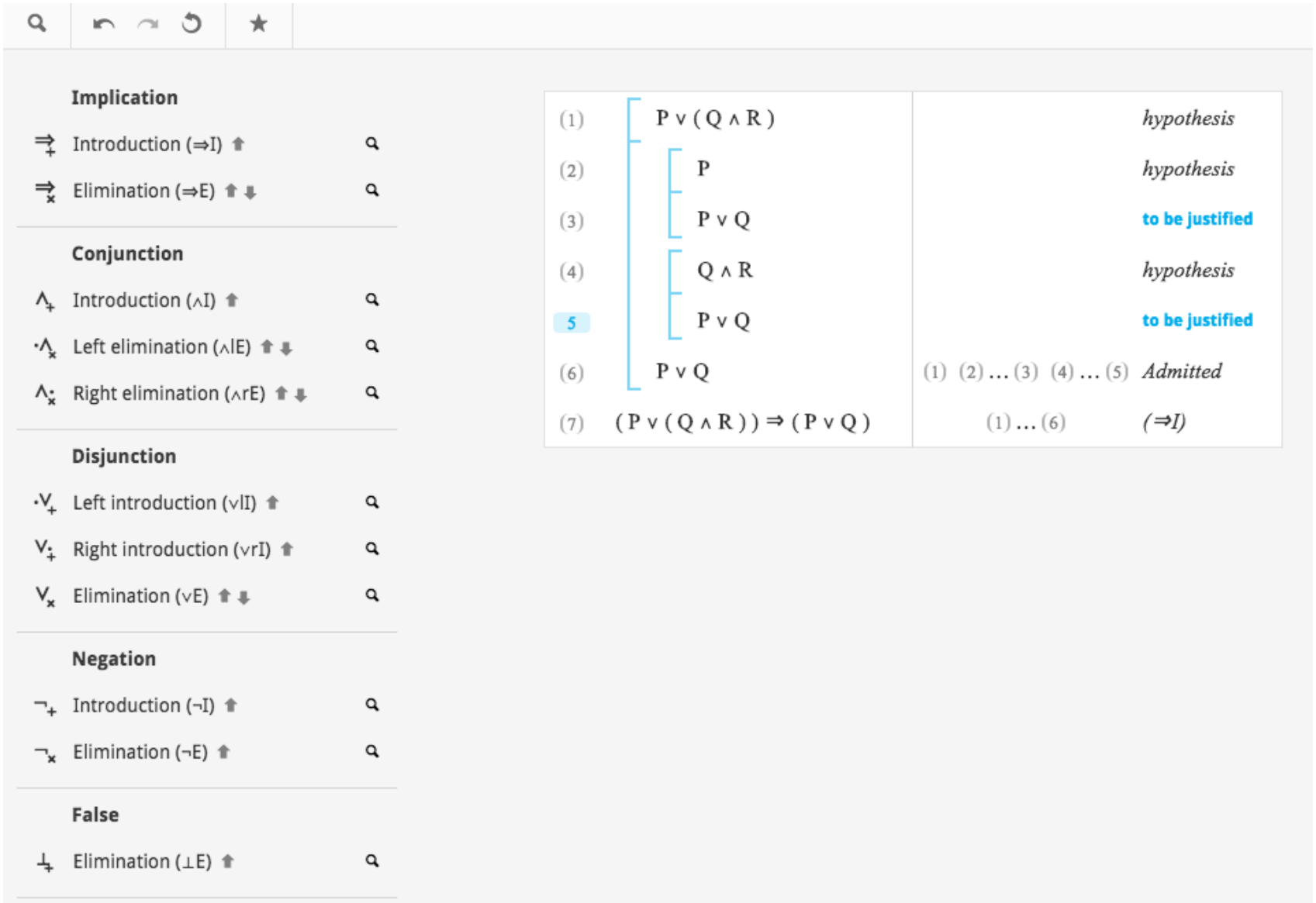
Edukera (Rognier and Duhamel)

- Web-application
- Coq is hidden inside the web page
- LCF style interaction + proof displayed in a pen and paper style.
- Some users in France (about 1000 students, 70k exercises)
- No textual input "proof by pointing", syntactically correct by construction (as using Scratch)
- Easy to learn using a tutorial
- Always correct applications of a logic rule
- Meta-variables

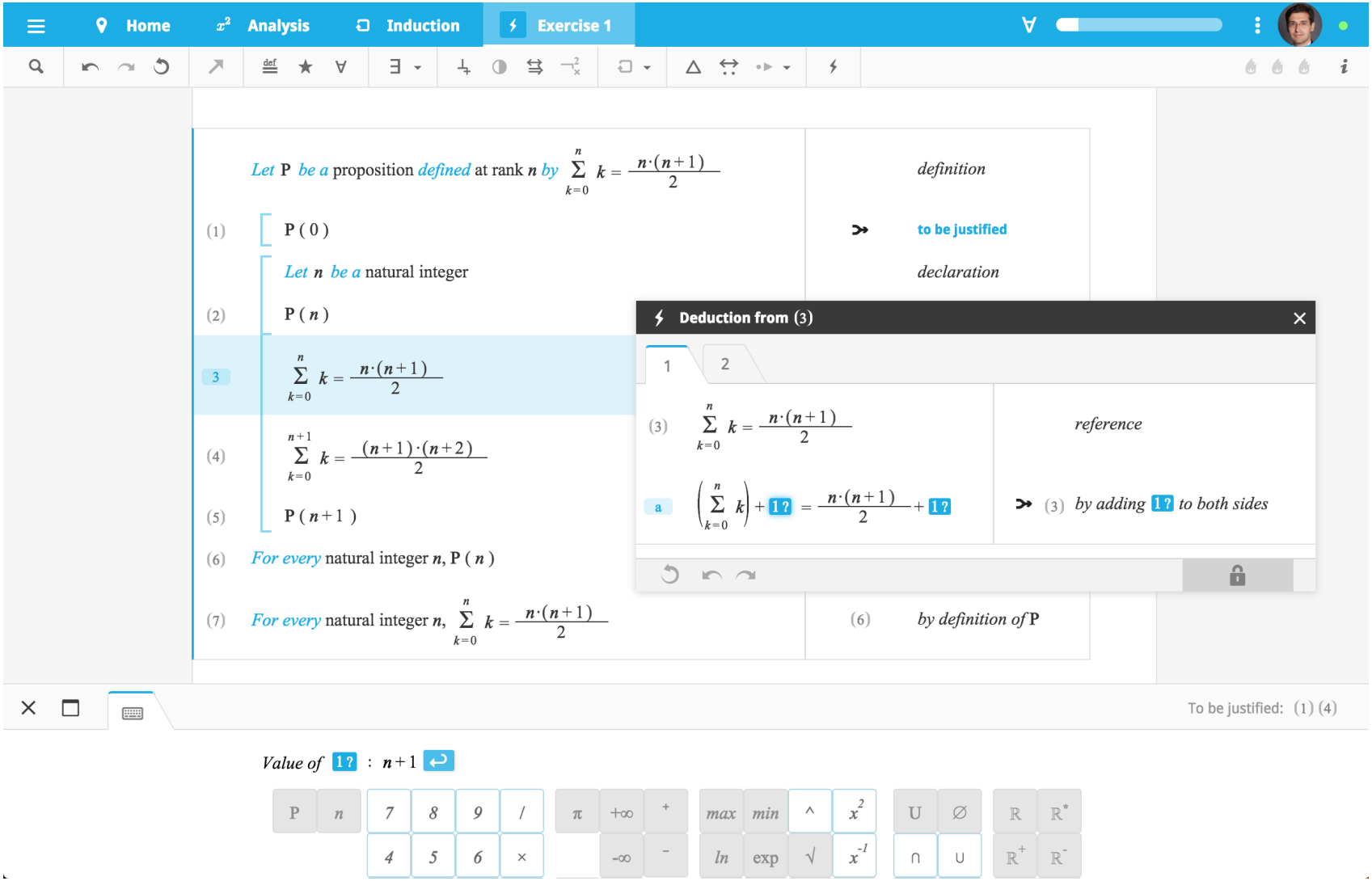
Two modes

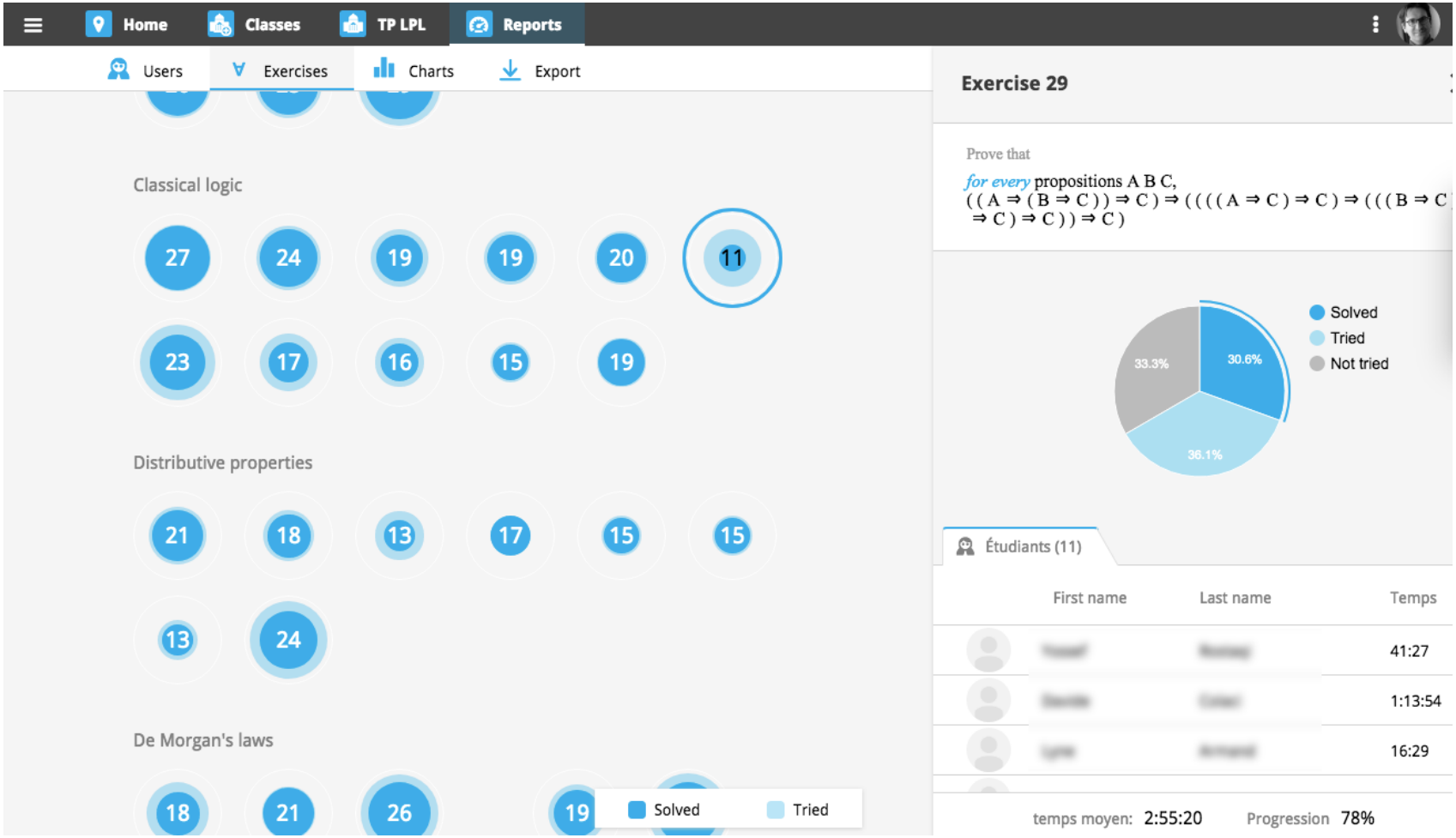
1. Logic
   - Use natural deduction rules.
   - Can display proof tree (Fitch's or Gentzen's style).
   - Backward reasoning
2. Maths
   - Forward/Backward reasoning.
   - Less fine-grained proof steps than in logic mode.

Edukera (logic mode)



Edukera (math mode)




Home
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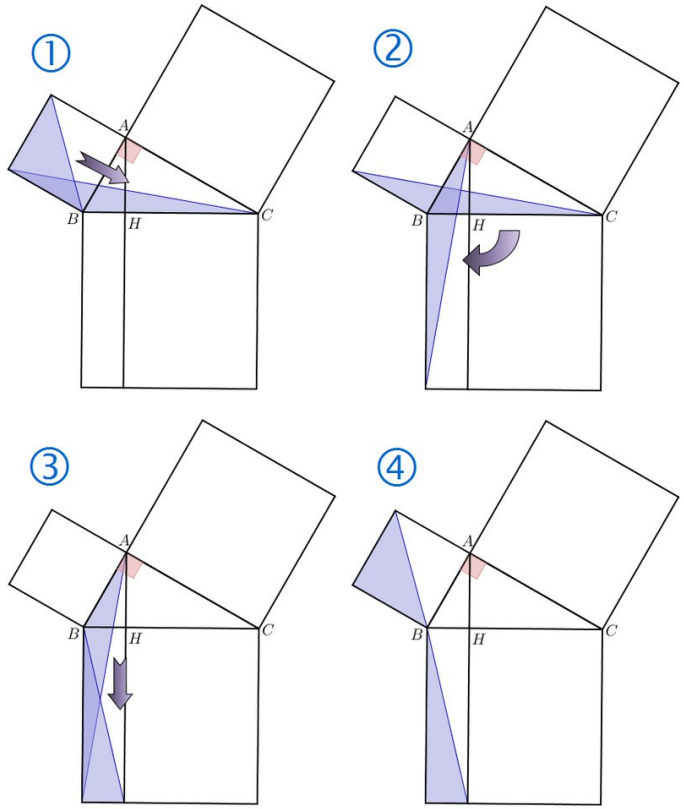
Solved
Tried
Exercise 29
Prove that
for every propositions A B C,
((A ⇒ (B ⇒ C)) ⇒ C) ⇒ ((((A ⇒ C) ⇒ C) ⇒ (((B ⇒ C) ⇒ C) ⇒ C)) ⇒ C)
Solved
Tried
Not tried
33.3%
30.6%
36.1%
Étudiants (11)
First name
Last name
Temps
41:27
1:13:54
16:29
temps moyen: 2:55:20
Progression 78%

Experiment at master lelvel

1. Choose a proof Pythagoras's theorem and propose a formalization.



Discussions :

- validity of proof
- proofs as explanation
- simplicity

- visual proofs
- circularity
- genericity (proof in a model, vs proof in every model)

Experiment with first-year students : Introduction to proof and proving

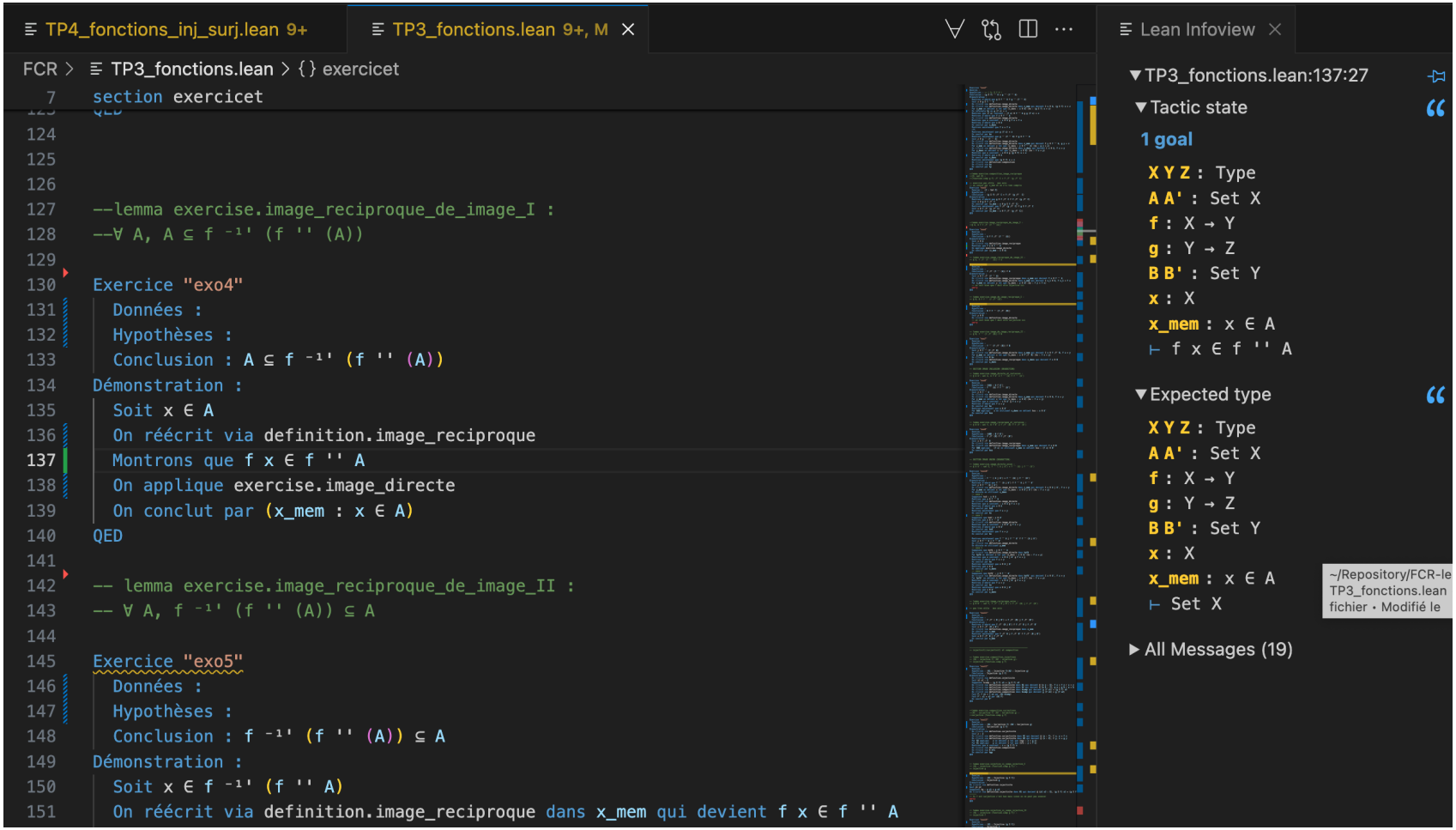
- L1 Maths and L1 CS, selected/not selected student.
- 7 groups.
- Mixed teaching team : high-school teachers, PhD students, lecturer in maths, lecturer in CS.
- Contents : vocabulary and structure of a mathematical document. Reasoning rules (without formality). Sets, Functions (direct image, inverse image, injective, surjective), Relations, induction.

Pre-experiment in Strasbourg in 2024

- Objective : introduction to proof
- 3 groups with Deaduction by Frédéric Le Roux
- 3 groups with Lean-Verbose by Patrick Massot
- On paper assessment.

Current work :

- Construction of a corpus of exercises (proofs, true/false, formalization).
- Improving software, bug reporting.
- Instrumentation of proof assistants to collect date.


TP4_fonctions_inj_surj.lean 9+
TP3_fonctions.lean 9+, M
FCR > TP3_fonctions.lean > { } exercicet
7 section exercicet
124
125
126
127 --lemma exercise.image_reciproque_de_image_I :
128 --∀ A, A ⊆ f ⁻¹' (f '' (A))
129
130 Exercice "exo4"
131 Données :
132 Hypothèses :
133 Conclusion : A ⊆ f ⁻¹' (f '' (A))
134 Démonstration :
135 Soit x ∈ A
136 On réécrit via definition.image_reciproque
137 Montrons que f x ∈ f '' A
138 On applique exercise.image_directe
139 On conclut par (x_mem : x ∈ A)
140 QED
141
142 -- lemma exercise.image_reciproque_de_image_II :
143 -- ∀ A, f ⁻¹' (f '' (A)) ⊆ A
144
145 Exercice "exo5"
146 Données :
147 Hypothèses :
148 Conclusion : f ⁻¹' (f '' (A)) ⊆ A
149 Démonstration :
150 Soit x ∈ f ⁻¹' (f '' A)
151 On réécrit via definition.image_reciproque dans x_mem qui devient f x ∈ f '' A
Lean Infoview
TP3_fonctions.lean:137:27
Tactic state
1 goal
X Y Z : Type
A A' : Set X
f : X → Y
g : Y → Z
B B' : Set Y
x : X
x_mem : x ∈ A
⊢ f x ∈ f '' A
Expected type
X Y Z : Type
A A' : Set X
f : X → Y
g : Y → Z
B B' : Set Y
x : X
x_mem : x ∈ A
⊢ Set X
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TP3_fonctions.lean
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```
Exercice "exo1"
  Données :
  Hypothèses :
  Conclusion : VraiFaux (∀ n : ℕ, (n = 0) ∨ (n = 1))
Démonstration :
  Montrons que ¬ ∀ n : ℕ, (n = 0) ∨ (n = 1)
  On pousse la négation
  Montrons que 3 convient : (3 ≠ 0) ∧ (3 ≠ 1)
  On calcule
QED

-- Plus petit que tous
-- Il existe m dans ℕ tel que pour tout n dans ℕ, m ≤ n.

Exercice "exo2"
  Données :
  Hypothèses :
  Conclusion : VraiFaux (∃ m : ℕ, (∀ n : ℕ, m ≤ n))
Démonstration :
  Montrons que ∃ m : ℕ, (∀ n : ℕ, m ≤ n )
  Montrons que 0 convient : ∀ (n : ℕ), 0 ≤ n
  Soit n
  On calcule
QED
```

In previous work [2] we studied one exercise using different PAs (Coq, Deaduction, Edukera, Lean (Verbose), Lurch) and we provided an *a priori* study of the potential impact of different features.

Exercise

Étant donnés trois ensembles A, B et C tels que $C \subseteq A$ et une fonction $f : A \to B$, montrer que si f est injective alors $f^{-1}(f(C)) = C$.

Some videos

2. Evmorfia Bartzia, Antoine Meyer et Julien Narboux (oct. 2022). "Proof assistants for undergraduate mathematics and computer science education : elements of a priori analysis". In : *INDRUM 2022 : Fourth conference of the International Network for Didactic Research in University Mathematics*. Sous la dir. de María Trigueros. Hanovre, Germany : Reinhard Hochmuth.

automatic unfolding of definitions

- fake success ? ($\rightarrow$ Lean Verbose / Coq Waterproof)
- real success since they get a proof ?

interaction style undo/redo, and automatic application of reasoning rules can lead to a trial/error strategy

Research in didactic ?

Very few results about proof assistants (Iannone, Hanna, Richard, Thoma).
Since 2024, ANR APPAM lead by Cécile Ouvrier-Buffet : didactics + maths+ computer science.
Study of student's difficulty while learning proof, impact of the use of proof assistants.

Some questions

- What competencies are exercised using PAs ?
- Does PAs improve habits ?
- Is the syntax a problem ?
- Syntax vs semantics ?
- How to transfer to proofs on paper ?
- What is the impact of foundations of PAs on student perception of proof ?

Do you want to know more ?

A thematic school : PAT

PAT 2023 : first thematic school about proof assistant for teaching, 38 participants.
PAT 2025 ? (project submitted to CNRS)

Workshops : ThEdu

ThEdu 2024, July 2, IJCAR, Nancy.

Bibliography

A Zotero group :
`www.zotero.org/groups/2621881/`